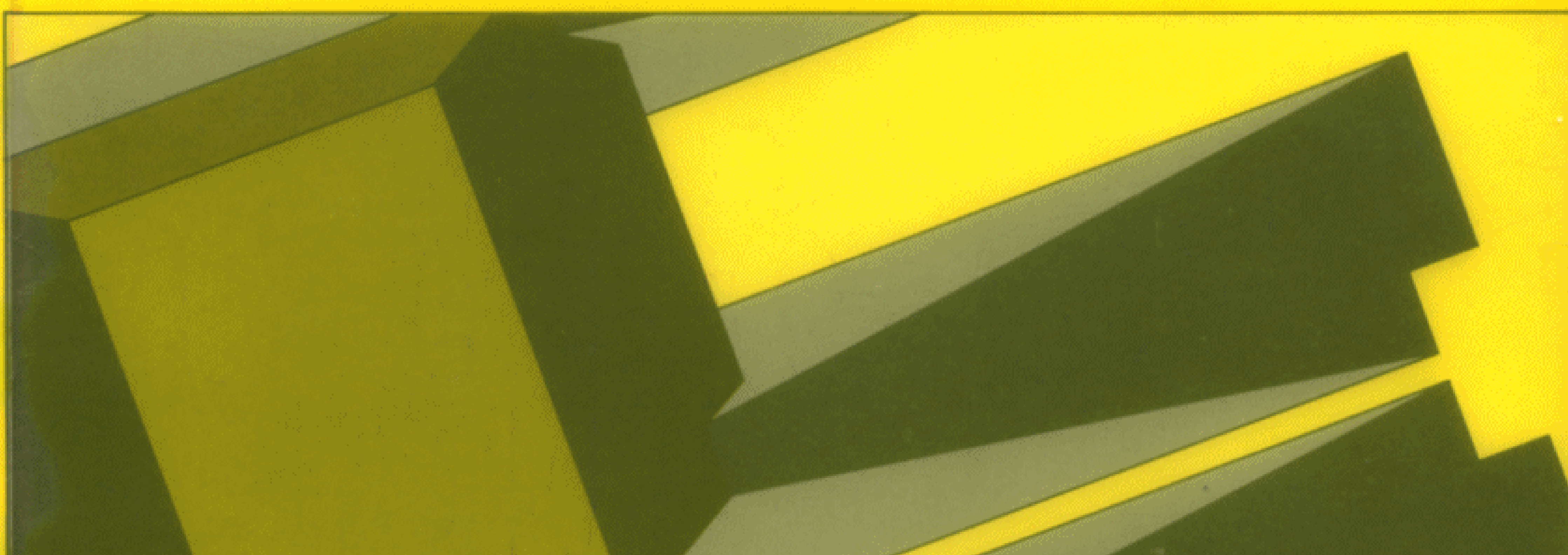
Engineering Design Guides 42

MECHANICAL SPRINGS

A A D Brown on behalf of
The Spring Research and Manufacturers' Association



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A. A. D. Brown B.Sc.
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Contents

Principal notation†

A	area
A'	area of inner ring
A''	area of outer ring
C_1, C_2 etc	parameters
D	(i) diameter, (ii) mean diameter of coil
D_a	diameter of arbor
D_b	diameter of bush
D_c	(i) inside diameter of case, (ii) twice radius of curvature
D_i	inside diameter
D_m	mean diameter
D_n	maximum outside diameter of coil in free state
D_o	outside diameter
D_u	inside diameter of unwound spring in case
D_w	outside diameter of wound spring on arbor
E	modulus of elasticity
F	force
G	shear modulus
I_1, I_2	parameters depending on shape and dimensions of the cross-section
J	axial moment of inertia
K	(i) correction factor, (ii) shape ratio parameter ($= m^{(0.014m + 0.46)}$), (iii) part of force equation for constant-load coiled beam, (iv) unit of comparison for leaf springs
K_c	$(1-\mu \tan \alpha)/(\tan \alpha+\mu)$
K_R	$(1+\mu \tan \alpha)/(\tan \alpha-\mu)$
K_r	stress concentration factor for rectangular-section material
K_s	a correction factor
K_1, K_2	stress factors
K_3, K_4, K_5	factors in design formulae for diaphragm springs
L	(i) length of moment arm of force P, (ii) solid length of spring
M	torque, modification constant
P	(i) applied force, (ii) load
P_a	axial load
Q	(i) axially applied load, (ii) displacing force
S	(i) spring rate, (ii) stress factor
V	volume
W	load at a deflection of $\delta/2$ from free
Z	modulus of a section
a	buckling factor
b	breadth, height of section
b_1	breadth at tip
b_0	breadth at root
c	$2r/d$
d	wire diameter
d_e	diameter of enlarged portion of bar
f	stress
f_D	design stress
f_b	bending stress
f_c	compressive stress
f_i	maximum stress at lower inner edge
f_t	tensile strength
f_u	maximum stress at upper surface
f_x	stress at distance x from fixed end of leaf
g	gravitational acceleration
h	height, radial depth of material
l	length
l_e	effective length
l_t	length of transition zone
m	shape ratio (breadth/thickness)
n	number
n_u	number of coils in unwound spring
n_w	number of coils in wound spring
p_i	initial tension in spring
r	radius
r_c	radius of stop
r_s	radius of spring
t	thickness
v	taper factor
α	(i) an angle, (ii) a factor
β	a factor
δ	deflection
δ'	deflection of inner ring
δ''	deflection of outer ring
δ_1	deflection of disc
δ_2	deflection of cantilever
δ_x	deflection at a distance x
η	natural frequency
η_2, η_3	Saint Venant coefficients
θ	an angle
λ	shape factor
σ	stress
μ	Poisson's ratio
τ	torsional stress
ν	natural frequency

†Owing to the large number of topics covered in this Engineering Design Guide, only a selection of the symbols used appear in this notation. The meanings of all the symbols employed in the Guide are given on their first appearance in the text.

Introduction

This Engineering Design Guide describes the design and application of, and gives manufacturing advice on, non-helical springs and springs stressed in bending, and thus deals with those springs not covered by *Helical springs* (The Spring Research and Manufacturers' Association 1974) in this series. None the less since *Helical springs* gives information on selection of materials, functional classification, special environments, and protective treatments, all of which are equally applicable to the springs described in the following pages, a knowledge of its contents is essential to the reader of the present Guide.

Since this Guide is produced in black and white, it is not possible to produce simple design-aid charts containing more than three variables and the attention of the reader is therefore drawn to *Spring Design Data Sheet* No 11 (see bibliography), which is produced in colour.

Torsion bars

If a straight bar of uniform cross-section is subjected to a torque, M, at its free end (see Fig. 1), that end will rotate through an angle of θ radians with respect to the fixed end given by

$$\theta = lM/I_1G,$$

where l is the length of bar between clamp and plane of torque, G is the shear modulus of the material, and I_1G is the torsional rigidity of the bar. The parameter I_1 depends upon the dimensions and shape of the cross-section.

The stress induced by the torque is zero at the centre of the area of the cross-section and increases as the distance from the centre increases so that it is only the stress, f, at the surface of the bar that is critical. This is given by

$$f = M/I_2,$$

where again the parameter I_2 depends upon the dimensions and shape of the cross-section.

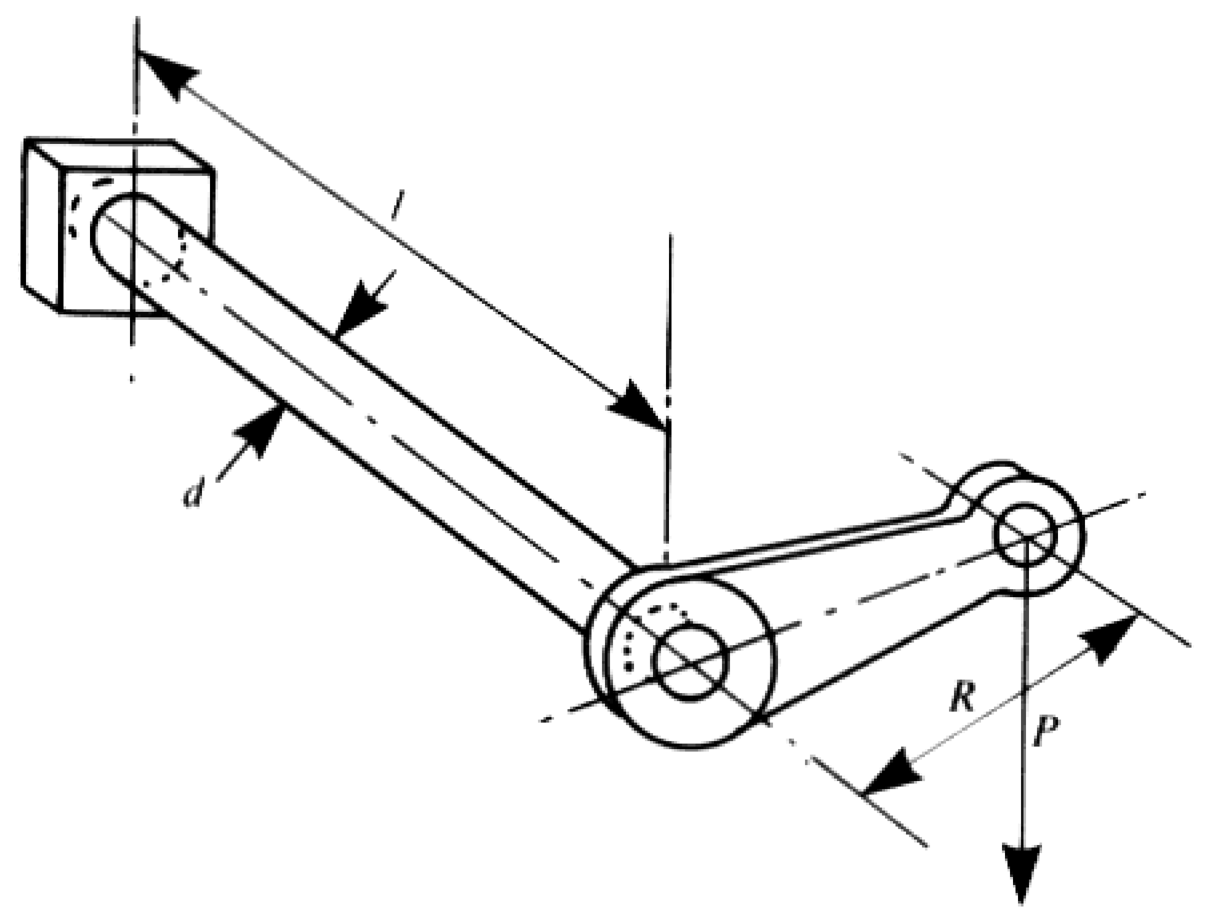

Fig. 1 Schematic torsion-bar assembly: P = applied load; R = horizontal distance between axis of bar and line of action of applied load

Circular cross-section

In the case of a bar of circular cross-section of diameter d, $I_1 = \pi d^4/32$ and $I_2 = \pi d^3/16$. Thus,

$$\theta = 32lM/\pi d^4G \text{ rad} \tag{1}$$

and

$$f = 16M/\pi d^3. \tag{2}$$

If these were the only considerations, the design of torsion bars would be a very simple matter. Unfortunately provision has to be made to clamp both ends of the bar in such a manner that the additional stresses imposed are not critical and the extra space needed is no greater than necessary. This is usually achieved by increasing the diameter of the bar for a short distance at each end so that the maximum skin stress occurs in the cylindrical section.

Figs 2 and 3 show typical tapers and additional cylindrical sections. The actual clamping is done on or through these sections by means of either splines or some form of cotter pin or wedge.

It is obvious that the tapered portion of the bar is less flexible than the cylindrical portion, length for length, so that it is necessary to calculate an effective length for substitution in eqn (1). This effective length will of course be shorter than the actual length of the bar between clamps.

The diameter, d_e, of the enlarged portion of the bar needs to be at least 40 per cent greater than d and this is usually achieved in one of two ways. Either a simple radius, r, is blended into the main section and sweeps up to meet the enlarged portion as in Fig. 2, or a straight taper of approximate included angle 30° is blended into the main section through a fillet radius of approximately $r = 1.4d$ (see Fig. 3). The latter is often termed the 'American' method, whereas the former generally finds favour in Europe. Both methods give very closely related results.

Referring to Fig. 2 it can be seen that the length of the transition zone, l_t is given by

$$l_t = \tfrac{1}{2}d\left(\frac{d_e}{d}-1\right)\Big/\sqrt{\left\{\left(\frac{4r}{d}\right)\left(\frac{d_e}{d}-1\right)-1\right\}}. \tag{3}$$

Assuming a constant diameter of d, the effective length, l_e, of the tapered portion is given by

$$l_e = vl_t,$$

where v is a factor obtained from Fig. 4. The equivalent 'American' length may be obtained from Fig. 5.

Consideration of the geometry in Fig. 3 (American method) shows that the wind-up of the tapered section is given by

$$\theta_t = (32Ml_t/\pi d^4G) \times \tfrac{1}{3}(x + x^2 + x^3),$$

where $x = d/d_e$. Reducing this to obtain the effective length gives

$$l_e = \tfrac{1}{3}l_t(x + x^2 + x^3).$$

In both the 'European' and the 'American' methods the *equivalent active length*, l, of the bar is given by, *length between tapers* + $2l_e$.

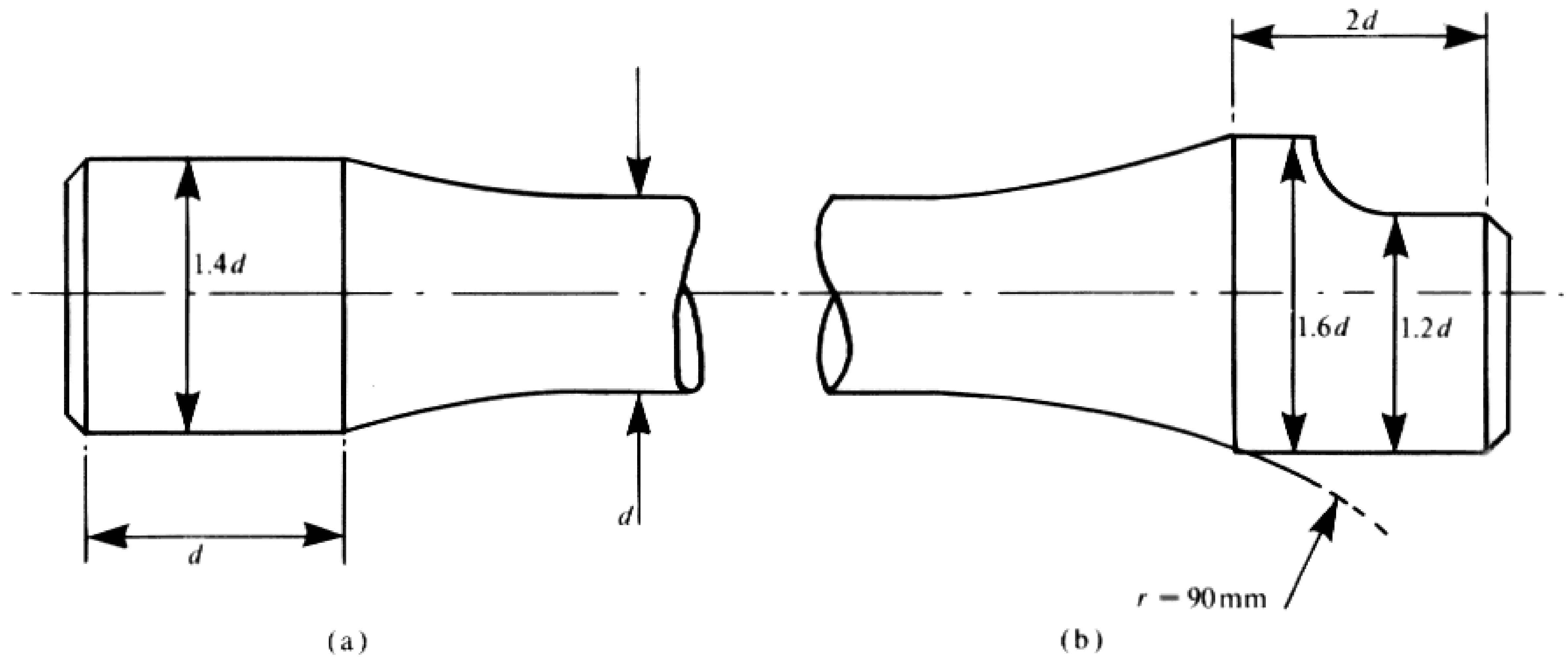

Fig. 2 Typical 'European' bar ends: (a) splined, (b) keyed

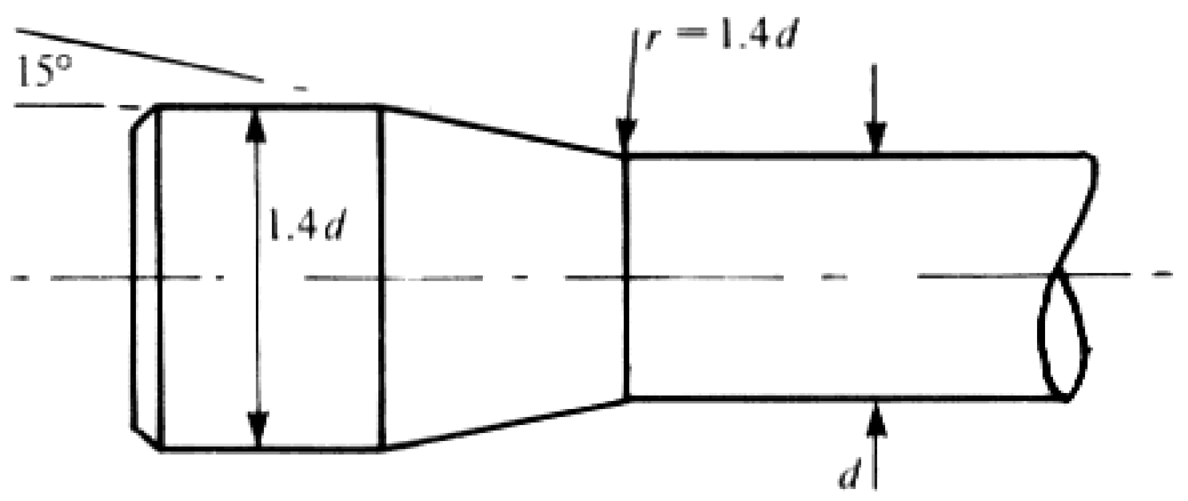

Fig. 3 Typical 'American' bar ends

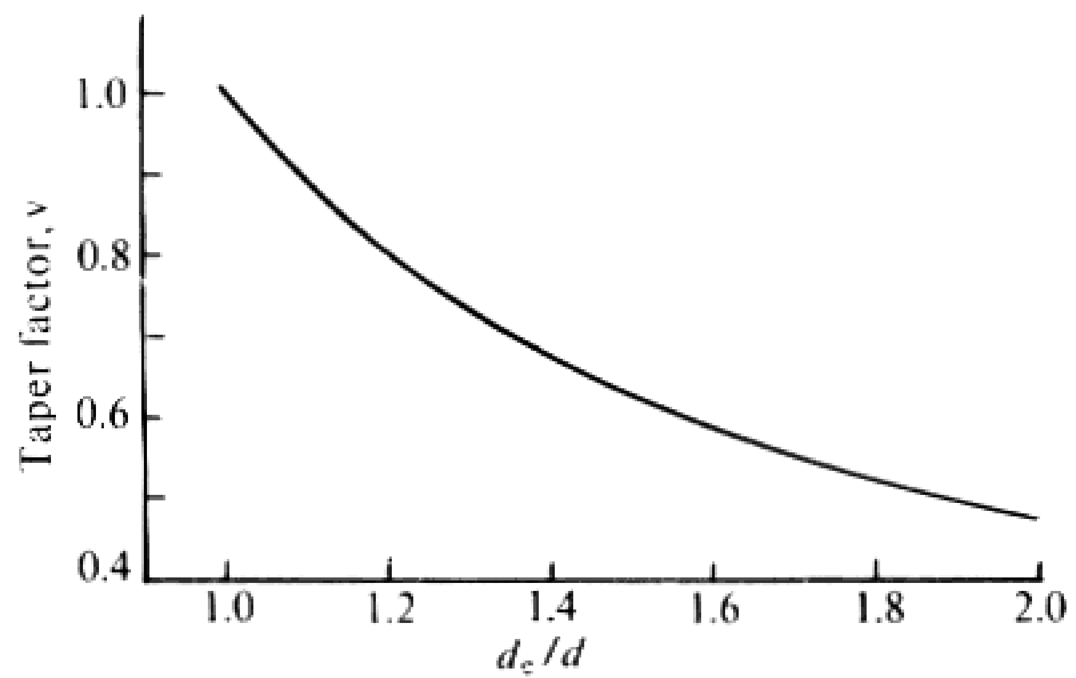

Fig. 4 Factor, *v*, to determine equivalent or active length of typical 'European' bar ends

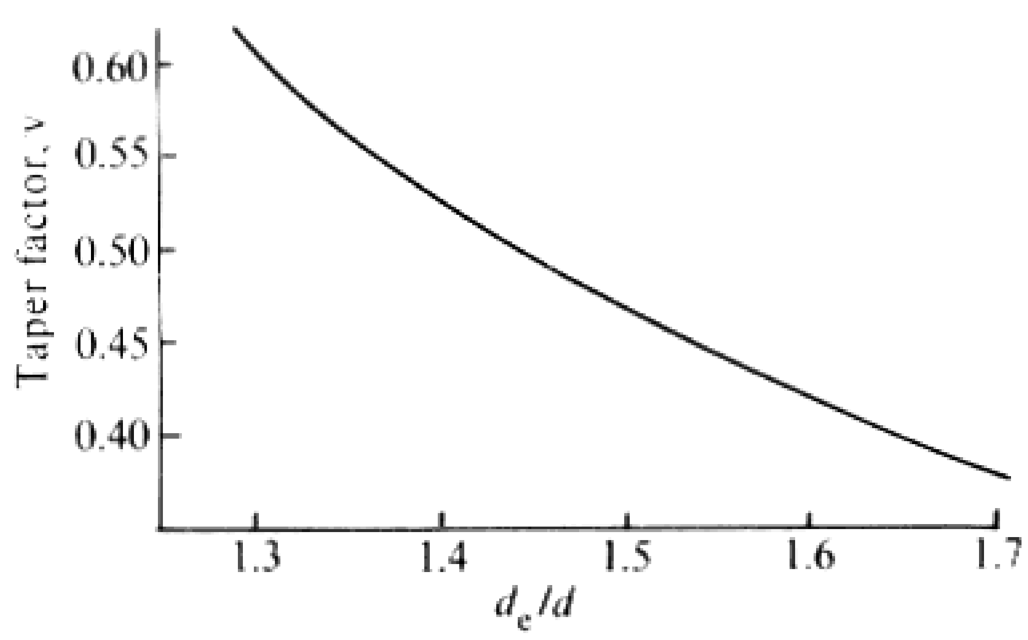

Fig. 5 Factor, *v*, to determine equivalent or active length of typical 'American' bar ends

It will be noticed that in both methods the design of the tapered section is governed by the diameter of the bar, the percentage effect of the taper varying inversely as the length of the cylindrical portion. If this effect is of the order of 1–2 per cent, it is probably compensated for by the flexibility of the mountings, but above this figure a further bar diameter modification is necessary.

Worked example

It is required to design a torsion bar using the 'European' method to give a torque of 120 000 N mm with a deflection of 0.3 rad, length between fixtures to be 400 mm, and maximum skin stress not to exceed 400 N/mm². To find diameter from eqn (2):

$$400 = 16 \times 120\,000/\pi d^3,$$

giving,

$$d^3 = 1528,$$

and

$$d = 11.5 \text{ mm}.$$

To find length from eqn (1), given that the shear modulus is 79 000 N/mm²:

$$0.3 = \frac{32l \times 120\,000}{\pi \times 11.5^4 \times 79\,000},$$

whence,

$$l = 354 \text{ mm}.$$

This shows that the bar is too short and therefore too small in diameter. The first adjustment is to increase l to 400 mm and resolve for d, whence,

$$d = 11.9 \text{ mm}.$$

If d_e/d is taken as 1.5, eqn (3) gives

$$l_t = (11.9/2)\,(0.5)\sqrt{\{(360/11.9)\,(0.5) - 1\}} = 11.2 \text{ mm}.$$

From Fig. 4, where v is taper factor,

$$v = 0.62.$$

Thus,

$$l_e = 0.62 \times 11.2 = 6.9 \text{ mm},$$

and the equivalent active length,

$$l = 400 - 2(11.2 - 6.9) = 391.4 \text{ mm}.$$

The strengthening effect is therefore 2.2 per cent and may be ignored.

Maximum skin stress $= 16 \times 120\,000/(\pi \times 11.9^3)$

$= 363$ N/mm².

Stability
Very long torsion bars that are subject to an axial compressive stress are liable to buckle and any error in co-axiality of the end fixings will increase this tendency. If the axially applied load is $\pm Q$, where the plus sign indicates a compressive load, the torque is M, the axial moment of inertia of the section is J, and the modulus of elasticity is E, then, to prevent buckling, it is essential that $(M/2EJ)^2 \pm (Q/EJ) < \pi^2/l^2$.

It will be seen that changing the sign of Q from + to − produces a more stable bar, so that one way of increasing stability is to subject the bar to a tensile load. If the bar is not constrained so that the end mountings are always co-axial, the control $2\pi^2/l^2$ becomes $a^2\pi^2/l^2$, where a is a buckling factor derived from whichever of the four Euler cases of buckling is relevant.

Splines
Where it may be necessary to make adjustment for set-down in service, it is usual to spline both ends of the bar so that one end may be advanced by a single tooth when necessary. This means, of course, that the setting, in degrees, should never be more than $360/2n$ out, where n is the number of teeth on each end.

Where it is necessary to be able to adjust to the next closer order of tolerances, the vernier adjustment method should be used. This is achieved by having one more tooth at one end of the bar than at the other. When both ends are advanced by a single tooth the differential adjustment is given, in degrees, by $360/n - 360/(n+1)$, where n is the number of teeth on the lesser end. Thus when $n=50$ the differential adjustment is 0.14°.

Where fine adjustment is not necessary and it has been established that no further set-down will occur after the first full-load deflection, the 'blocked-spline' method may be used. This consists of leaving one tooth uncut at each end of the bar itself and a corresponding keyway on the female portions of the fittings. This ensures that the bar can only be assembled in one position and that the pre-load is reasonably constant.

Permissible stresses
For static applications a bar of low-alloy spring steel may be stressed to approximately 700 N/mm², although if it is fully pre-stressed this value may be raised to 930 N/mm². Care must be taken to ensure that in the latter case the bar is pre-stressed in the direction of loading, and as a precautionary measure an arrow indicating the hand of pre-stressing should be stamped on each end of the bar.

The fatigue properties of torsion bars that are always stressed in the same direction and oscillate about a mean stress which is higher than half the stress range may be enhanced by surface treatment, the best of which appears to be a fine surface grind followed by all over shot peening. For bars exceeding 26 mm diameter there is an exponential falling off of fatigue resistance as the diameter increases. Typical fatigue diagrams for silico-manganese bars 25–62.5 mm in diameter in the fine-ground, and ground and shot-peened states are shown in Fig. 6. Bars smaller than 25 mm have almost identical properties to those of 25 mm. Under these conditions a life of 50 000 full bump operations should be anticipated.

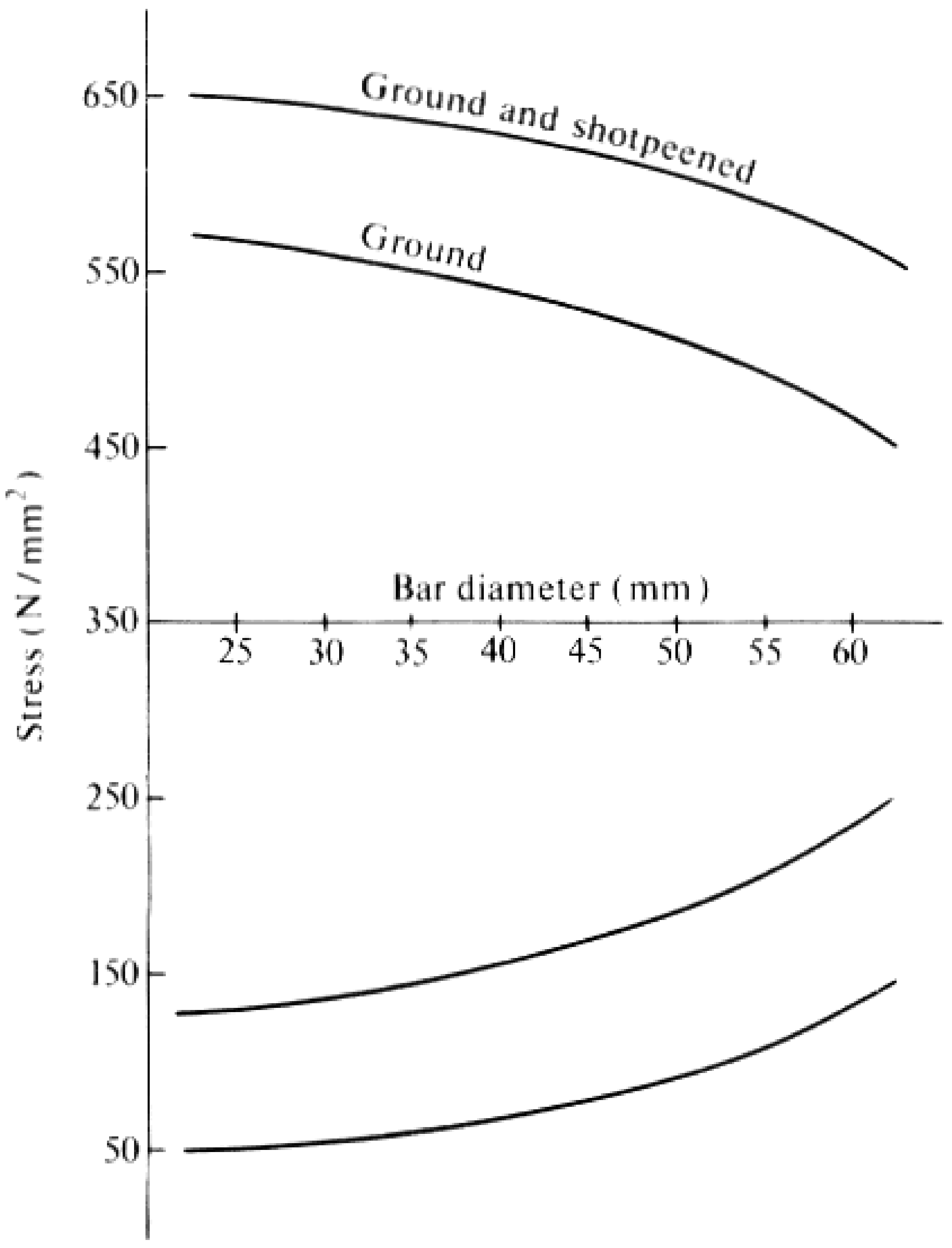

Fig. 6 Fatigue diagram for bar stressed in one direction about a mean of 350 N/mm²

Rectangular cross-section
In the case of a bar of rectangular cross-section ($b \times t$), equations (1) and (2) become respectively:

$$\theta = lM/\eta_3 t^3 bG \text{ rad} \tag{4}$$

and

$$f = M/\eta_2 t^2 b, \tag{5}$$

where b is the long side, t is the short side, and η_2 and η_3 are the *Saint Venant* coefficients. Values of these coefficients, which are also used in the calculation of helical springs of rectangular section stressed in torsion, may be read off from Fig. 7.

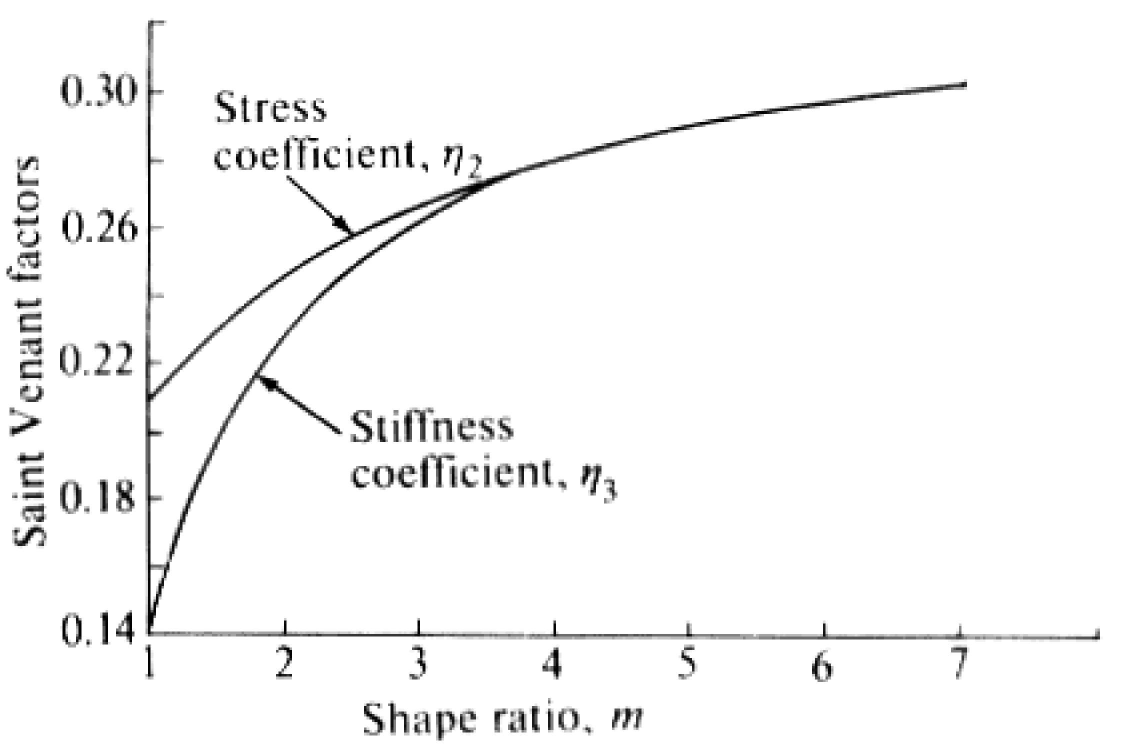

Fig. 7 Saint Venant design factors for rectangular sections; for values of $m > 5$, $\eta_2 = \eta_3 = (m - 0.63)/3m$, where $m = b/t$

Comparison with round section
If an attempt is made to satisfy the requirements specified for the round-section bar while using the simplest form of rectangular section (i.e. square) then, from eqn (5),

$$400 = 120000/0.208b^3,$$

whence

$$b^3 = 120\,000/0.208 \times 400,$$

and

$$b = 11.25 \text{ mm};$$

and from eqn (4)

$$l = 0.3 \times 0.14 \times 11.25^4 \times 79\,000/120\,000$$
$$= 445 \text{ mm}.$$

Thus at the same stress it is impossible to replace a round-section bar by a square-section bar of the same length. In order to accommodate the bar in a length of 400 mm its section must be reduced to 10.9 mm with a consequent increase in stress to 415 N/mm², i.e. an increase of 16.5 per cent.

Let us now see what happens when there is a slight departure from the square section to a rectangular section of shape ratio, $m = b/t = 1.4$. Simple arithmetical exploration leads to a bar of dimensions 14.76 mm × 9.84 mm capable of sustaining a torque of 120 000 N mm at a stress of 356 N/mm² (the same as the round-section bar). Using $\eta_3 = 0.196$, then $l = 557$ mm, which is even longer than the bar of square section. The appropriate dimensions for accommodation within a length of 400 mm are $b = 13.2$, $t = 8.8$, from which $f = 500$ N/mm², representing an increase of 40 per cent over the round section.

Initially it would therefore appear that any rectangular-section torsion bar may be replaced by one of circular section in which the material is used more efficiently, so that the question may be raised as to whether it is ever necessary to use any section other than circular. The answer lies principally in the fact that a rectangular-section bar may be located in a matching rectangular hole without the costly machining to produce the ends necessary on a bar of circular section. If the quantity is sufficiently high, bar may be purchased in the precision-rolled condition, cut to length, hardened, and jig-tempered to ensure adequate straightness. As with round-section bars, the fatigue life of those of rectangular section is increased by pre-stressing, and once again care must be taken to identify the mode in which the bar has been pre-stressed.

Laminated torsion bars
If the specified torque, angular deflection, and available length are such that the maximum permitted stress is always exceeded, then resort must be had to the device used under similar circumstances when designing leaf-springs in bend, i.e. the spring must be built up of leaves of sufficiently small torsional rigidity to permit of the desired deflection while using the appropriate number of leaves. A spring of this kind is known as a *laminated torsion bar* and has been successfully used in a number of applications for some considerable time.

The author is unaware of any successful theoretical analysis of the load–deflection characteristic of such a spring and, from consideration of the following factors it appears that such a general solution does not exist.

1. With a spring having an odd number of leaves only the central leaf behaves in the same manner as a circular-section bar when loaded in the same mode, while the non-central leaves attempt to take up a position where the centre-line of each is a helix whose form is dictated by its distance from the centre line of the pack.
2. As the bar twists those portions retained within the anchorage tend to fan out, much in the manner of a 'hand' of playing cards held normally. Any restriction of this tendency imposes additional stresses at the point of anchorage, with a resulting proportional increase in torque.
3. In taking up the position of a helix, the active portion of the spring shortens and a compensating length is withdrawn from the anchorage, so that each pair of springs, numbering from the inside outwards, moves by an increasing amount.
4. As the shape ratio, m, increases above single figures the load–deflection characteristic of even a single bar departs very rapidly from the straight. A typical curve obtained in practice for a bar 250 mm long and 13.5 mm × 0.45 mm in section is shown as curve (a) in Fig. 8, and it will be observed that at 100° angular deflection the measured torque is more than double the calculated value.

None the less, an intensive study of the problem during the immediate post-war period led to the following practical solution. Provided that

(1) the shape ratio b/t is kept between 5 and 10,
(2) the number of leaves equals the shape ratio (i.e. $n = m$, so that the pack has a square cross-section),

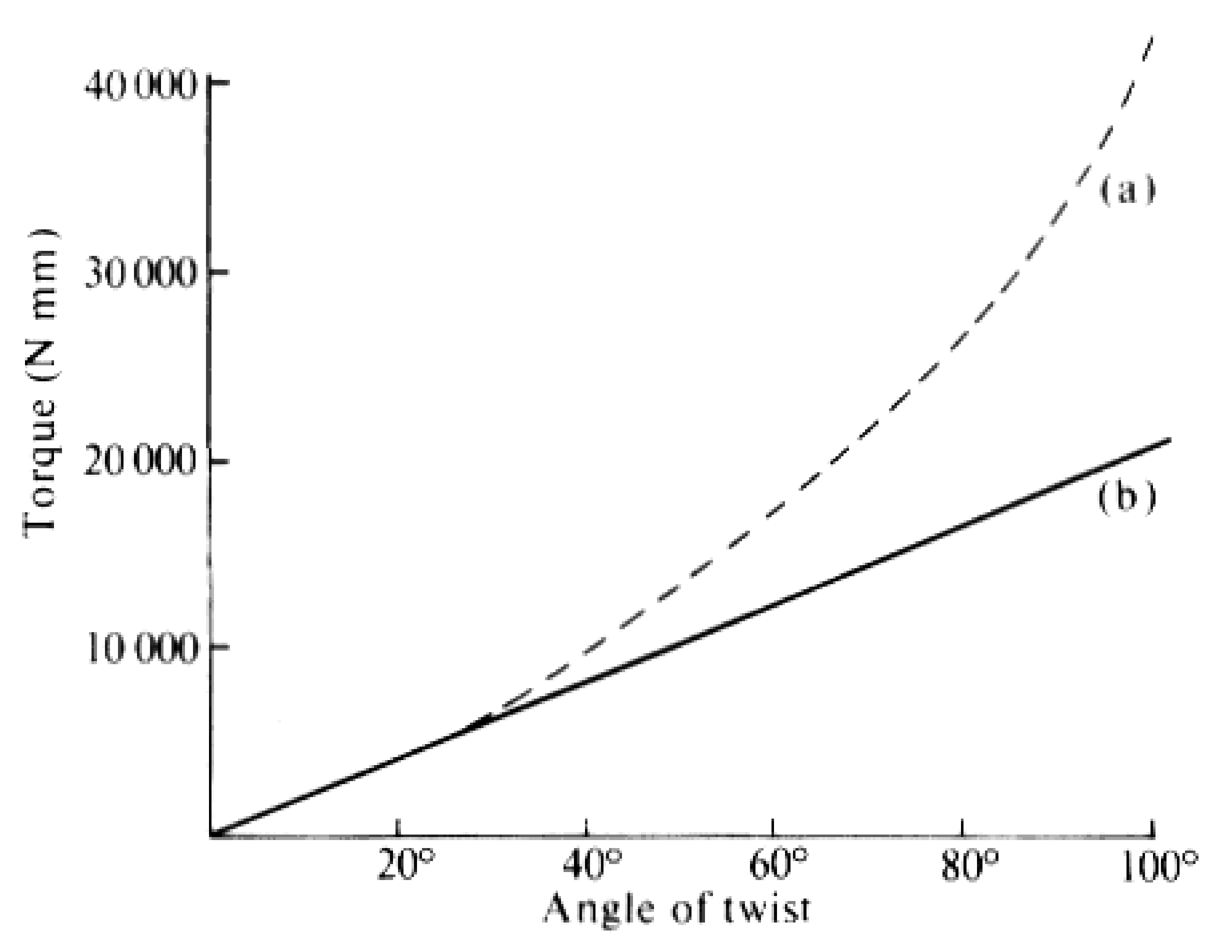

Fig. 8 Characteristic of single rectangular bar in torsion: (a) measured torque, (b) torque calculated from $m = \eta b^3 t G\theta/L$

(3) the theoretical stress in the central bar (calculated as a single rod in torsion without helical distortion) is kept below 500 N/mm², and

(4) the length of the bar is such that the maximum angular deflection is approximately 90°,

then torque,

$$M = 11\,320 n b^2 t^2 \theta / l m^{(0.014m + 0.46)} \text{ N mm.}$$

This may be rewritten as follows:

$$M = 11\,320 n b^2 t^2 \theta / lK, \qquad (6)$$

where $K = m^{(0.014m + 0.46)}$. Since $4 < m < 11$ and m is always a whole number the following simple table facilitates design calculations:

m	5	6	7	8	9	10
K	2.35	2.65	2.96	3.29	3.62	3.98.

Stress is given by

$$f = 30M/n(n - 0.63)t^3 \text{ N/mm}^2. \qquad (7)$$

Further work by the author is described in the *Appendix* (p. 42).

Worked example

It is required to design a laminated torsion bar of minimum cross-section and length 750 mm to give a torque of 140 000 N mm ± 5 per cent with an angular movement of 1.4 rad and a maximum stress of 500 N/mm². Unless previous experience dictates otherwise, choose m near the middle of the range. Thus if $m = 7$, then $K = 2.96$. From eqn (7),

$$t^3 = \frac{3 \times 140\,000}{7 \times 6.37 \times 500} = 18.84,$$

whence,

$$t = 2.65 \text{ and } b = 18.55.$$

From eqn (6) torque at 1.4 rad,

$$M = \frac{11\,320 \times 7 \times 18.55^2 \times 2.65^2 \times 1.4}{2.96 \times 750} = 120\,750 \text{ N mm.}$$

This bar is too weak, and one with fewer leaves is required. Since the torque is within about 15 per cent of the requirement, then the next number down (i.e. 6) is tested as follows:

$$t^3 = \frac{3 \times 140\,000}{6 \times 5.37 \times 500} = 26.07$$

whence,

$$t = 2.96 \text{ and } b = 17.76.$$

For the sake of possible material availability try $t = 3$, $b = 18$, giving a torque at 1.4 rad,

$$M = \frac{11\,320 \times 6 \times 18^2 \times 3^2 \times 1.4}{2.65 \times 750} = 139\,500 \text{ N mm.}$$

This result is within tolerance and the stress is then calculated:

$$f = \frac{3 \times 140\,000}{6 \times 5.37 \times 27} = 483 \text{ N/mm}^2,$$

which is an acceptable result.

The design is therefore as follows: 6 leaves each of 3 mm × 18 mm, × 750 mm free length.

It is customary to leave a minimum length equal to the bar width for fixing at each end of the bar, so that there should be a minimum overall length of 750 + (2 × 18) = 786 mm.

Torsion nests

If it is necessary to design a torsion device that is virtually free from friction in a space that prohibits the use of a single torsion bar, a neat but very expensive method is available in the *torsion nest*. This consists of a number of identical cylindrical torsion bars arranged symmetrically on the circumference of a circle, their ends located in holes in end plates so that the spring has the appearance of a cage. When one plate is rotated with respect to the other the individual bars are deformed in pure torsion and pure bend, each of which may be treated separately before it is compounded into a single set of design particulars.

Suppose there are n such bars, each of diameter d and length l, arranged uniformly around a circle of diameter D. Then

$$\theta = \frac{32lM}{(\pi n d^4 G)\,\{1 + 3(ED^2/Gl^2)\}} \text{ rad,}$$

where E is the modulus of elasticity and G the shear modulus. The maximum torsional stress, f_t, and maximum bending stress, f_b are expressed by $f_t = dG\theta/2l$ and $f_b = 3dDE\theta/2l^2$.

If these are combined to give

$$f_c = \sqrt{(3f_t^2 + f_b^2)},$$

then f_c may be considered as a tensile stress and related to the tensile properties of the material.

Materials and applications

All the recognized spring materials may be used for the manufacture of torsion bars, the choice depending upon size, type of end fitting and environment.

Vehicle suspension

With the development of independent front-wheel suspension for road vehicles the high volume efficiency of the torsion bar has made this system popular throughout the industry. The rotational movement of the spring must, of course, be translated into a vertical movement and this is usually achieved by means of a swinging arm which forms part of a parallel-link system (Fig. 9).

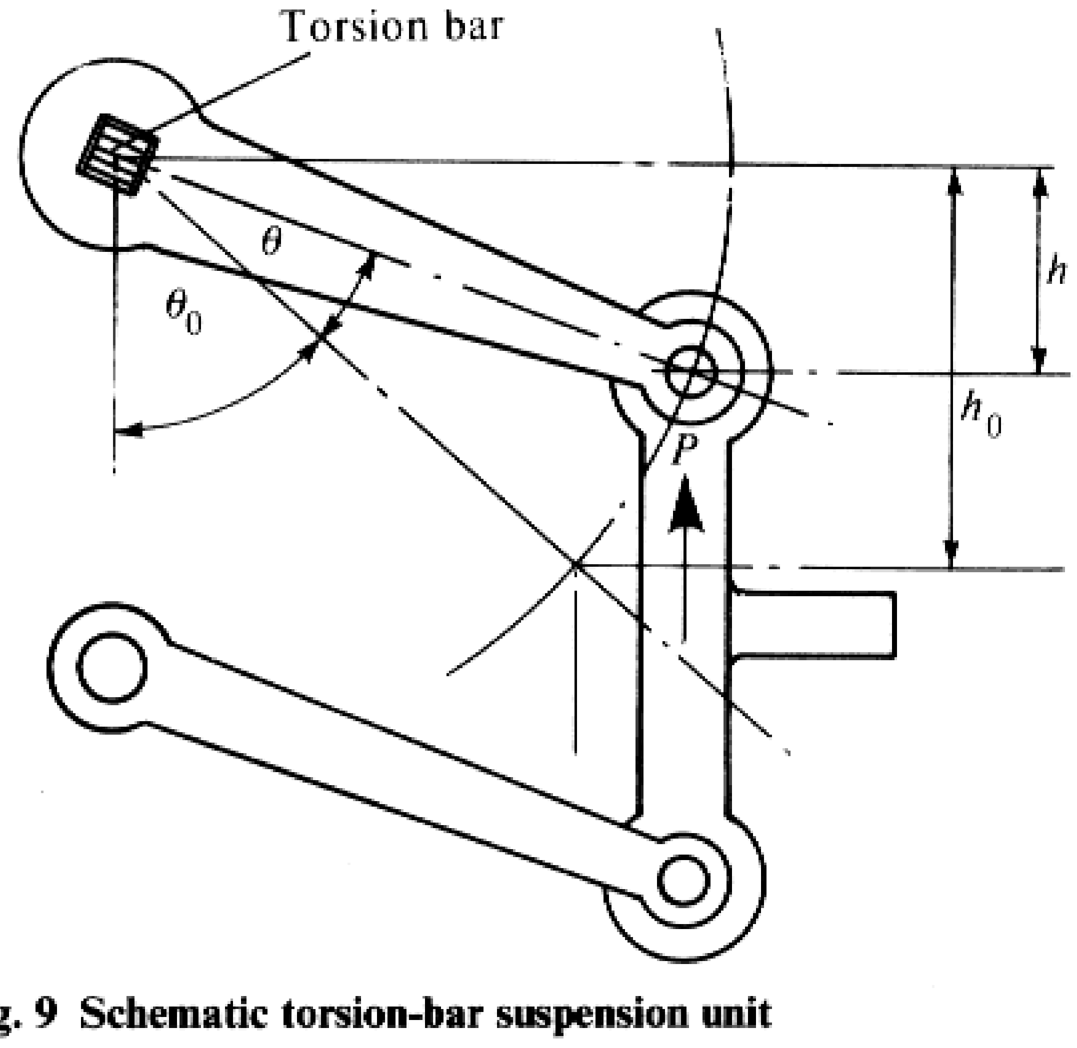

Fig. 9 Schematic torsion-bar suspension unit

In the free position the swinging arm makes an angle of θ_0 with the vertical. In the static position an upward force P moves the arm through an angle of θ, while the distance that the point of application of the load is below the horizontal through the centre of the bar moves from h_0 to h. Thus the basic equations governing the layout are:

$$h_0/r = \cos \theta_0,$$

$$h/r = \cos (\theta_0 + \theta),$$

$$\sin (\theta_0 + \theta) = \sqrt{\{(r^2 - h^2)/r^2\}},$$

and

$$M_\theta = Pr \sin (\theta_0 + \theta),$$

where r is the length of the lever arm.

It will be appreciated that the vertical rate of such a system is not constant and if M_θ/θ is denoted as S_θ then the rate of the static position, S, can be calculated:

$$S = \frac{S_\theta}{r^2}\left\{\frac{1 - \{\theta/\sin(\theta_0 + \theta)\}\{\cos(\theta_0 + \theta)\}}{\sin^2(\theta_0 + \theta)}\right\},$$

and since the natural frequency of an oscillating system is given by $(\text{rate/mass})^{\frac{1}{2}}/2\pi$ per second, the frequency may be calculated.

If the system is applied to the front suspension of a road vehicle the natural frequency should be kept between 100/min and 130/min, whereas for rear suspensions the accepted limits are 70 and 100.

If it is desired to obtain torsion-bar suspension springs for a road vehicle, the following data should be supplied to the designer:

(1) position on vehicle (front or rear);
(2) static load on wheel in normally laden condition;
(3) length of swinging arm;
(4) vertical travel from 'free' to static position;
(5) maximum additional vertical travel under 'bump' conditions;
(6) distance between plane of torque and 'anchor';
(7) available space in the other planes, and
(8) natural frequency limits if different from standard.

The designer will then design the simplest form of bar to meet the specification and will only proceed to irregular shapes or lamination if necessary.

Door closers

The generic term *door closers* covers a whole family of springs producing relative movement between two component parts of a mechanism that are hinged together. The simplest form, from which the name is derived, consists of an elongated **Z** formed from hard-drawn wire and having a short length at each end turned out at right angles. One turned-out end is fixed to the edge of the door adjacent to the frame, and the other end is twisted until sufficient torque is produced to close the door. The second end is then attached to the door frame. The versatility of this spring is accounted for by the fact that the ratio of length to diameter is several hundreds, allowing a number of complete rotations of one end before the stress becomes prohibitive. Between this simple wire form and the multileaf torsion bar lie the multitude of springs that must be designed to meet specific requirements. These should be specified quoting

(1) available length;
(2) torque necessary to *complete* the closing movement (the torque while the hinge is open will always exceed this);
(3) the angular movement from open to closed;
(4) the approximate frequency of operation, and
(5) the expected life in cycles.

The torque necessary to close the hinge may be measured by attaching a spring balance at a fixed distance from the hinge pin and pulling at right angles to the radius until the hinge is fully closed. The travel should be as short as possible to avoid 'momentum' effects which may not be present in practice.

The constant-load coiled beam

Because of the complicated nature of the movement under load of the spring shown in Fig. 10, it is very difficult to find a name for it that is truly descriptive. The main body of the spring is a closely wound helix (with or without initial tension), while the operating arms perform the double function of increasing the lever ratio and at the same time increasing the deflection by rotation about their own instantaneous centres. Perhaps the popular American name 'Flexator' is as good a single-word description as any.

Fig. 10 shows the spring in the free and the loaded positions, where P is the applied force in newtons, n is the number of coils in the helical spring, d is the diameter of the wire (mm), D is the mean diameter of the coils (mm), l is the length of the arm (mm) measured to the centre of the bearing eye, α is the initial angle between the arm (measured through the

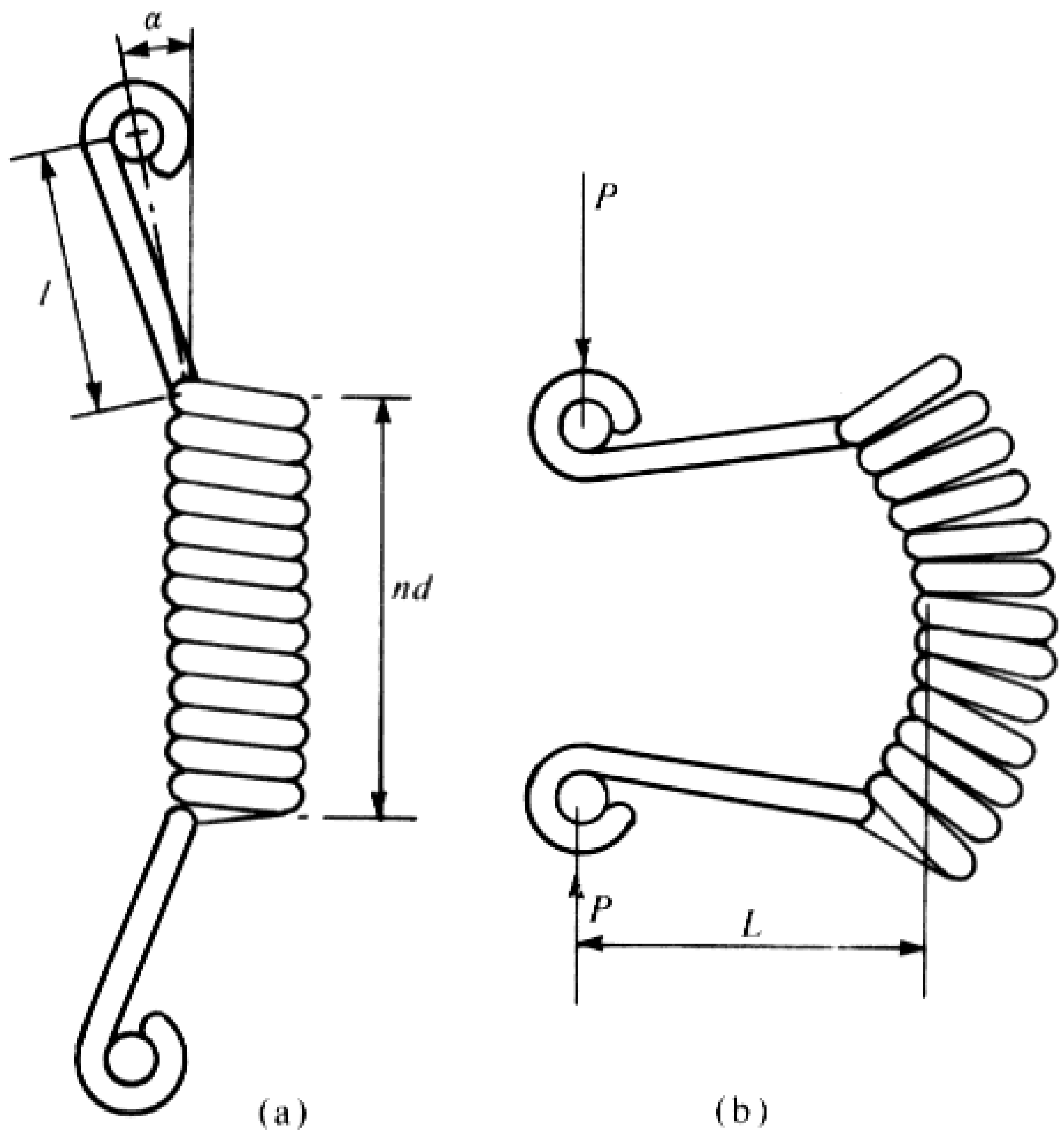

Fig. 10 Typical constant-load coiled beam: (a) free, (b) loaded

centre of the bearing eye) and the axis of the spring, θ is the angle of deflection of the arm under load,† L is the length of the moment arm of force P (mm) measured between the line of action of P and the point of contact of the remotest pair of adjacent coils, and p_i is the initial tension in the spring measured in newtons.

In practice it is usual to pivot both eyes on free-moving pins or bearings and constrain one to move towards the other on a line parallel to the original axis of the spring, the shortening of the distance between the eye centres being taken as the spring deflection (δ mm).

Load–deflection characteristics

The applied torque is made up of two parts: (1) $\theta G d^4/8nD$, where G is the shear modulus, resulting from the flexing of the spring without initial tension, and (2) $p_i D/2$, resulting from the initial tension.

The effective leverage is given by $l \sin(\alpha+\theta) + (nd/2\theta)\{\cos(\theta/2) - \cos\theta\}$, and if the sum of (1) and (2) is divided by this leverage then the force parallel to the original spring axis, is obtained. For ease of computation K, which equals $\{\cos(\theta/2) - \cos\theta\}/2\theta$ for values of θ up to 1.4 rad, is given in Fig. 11, and the equation for force becomes

$$P = \{(\theta G d^4/8nD) + (p_i D/2)\} \div \{l \sin(\alpha+\theta) + Knd\}. \quad (8)$$

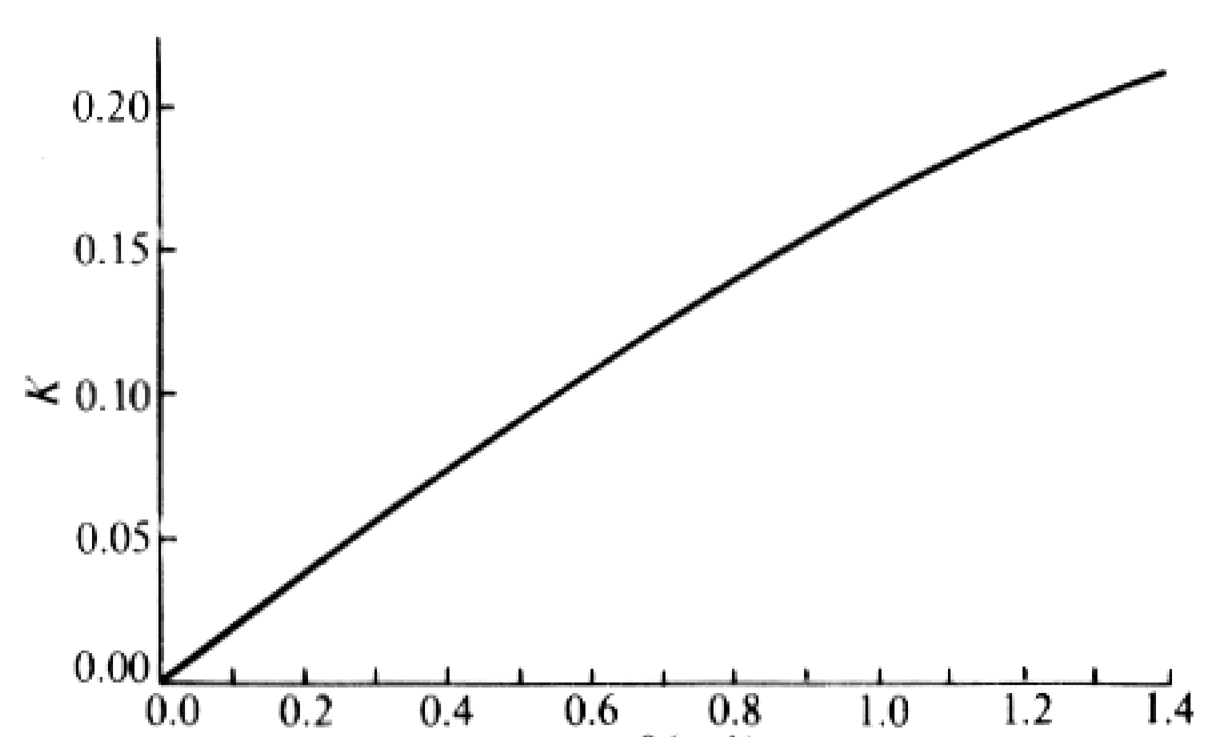

Fig. 11 Values of $K = \{\cos(\theta/2) - \cos\theta\}/2\theta$

Deflection

The amount by which one pivot centre approaches the other is given by

$$\delta = nd\{1 - (\sin\theta/\theta)\} + 2l\{\cos\alpha - \cos(\alpha+\theta)\}. \quad (9)$$

In this equation the first term represents the movement caused by flexing of the helical part of the spring, while the second represents that caused by rotation of the arms about their instantaneous centres.

Because of the complexity of an expression containing four variable terms, it is simpler to

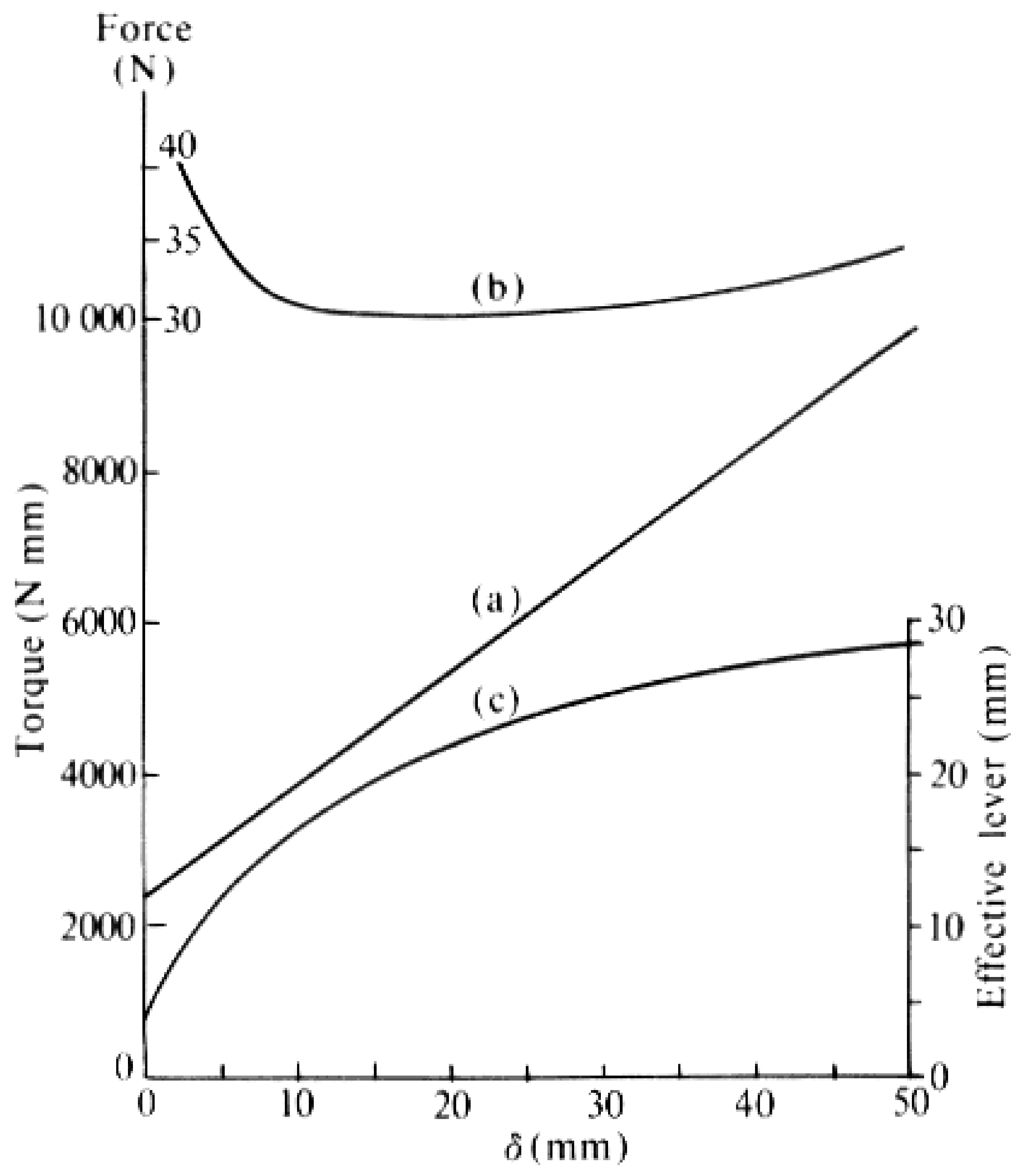

Fig. 12 Curves for a typical beam: (a) force, (b) torque, (c) effective leverage

appreciate the significance of each by considering a standard spring and investigating the effect of varying each term in turn.

If the standard chosen is $n=20$, $d=0.2$ mm, $D=15$ mm, $p_i=30$ N, $\alpha=0.15$ rad, $l=20$ mm, and $G=79\,000$ N/mm², then, $P = (527\theta + 225)/\{20 \sin(0.15+\theta) + 40K\}$, and if the numerator, the denominator, and P are plotted on the same graph (see Fig. 12), then it can be seen that in this instance the spring rate, which is initially negative, is approximately zero over a considerable range before finally increasing.

For most applications the designer is interested only in the constant-load zone, and here investigation is confined to the effect of the changes of variables in this zone.

Initial tension

Fig. 13 shows the effect of increasing initial tension in the spring from zero to 30 N and then to 50 N.

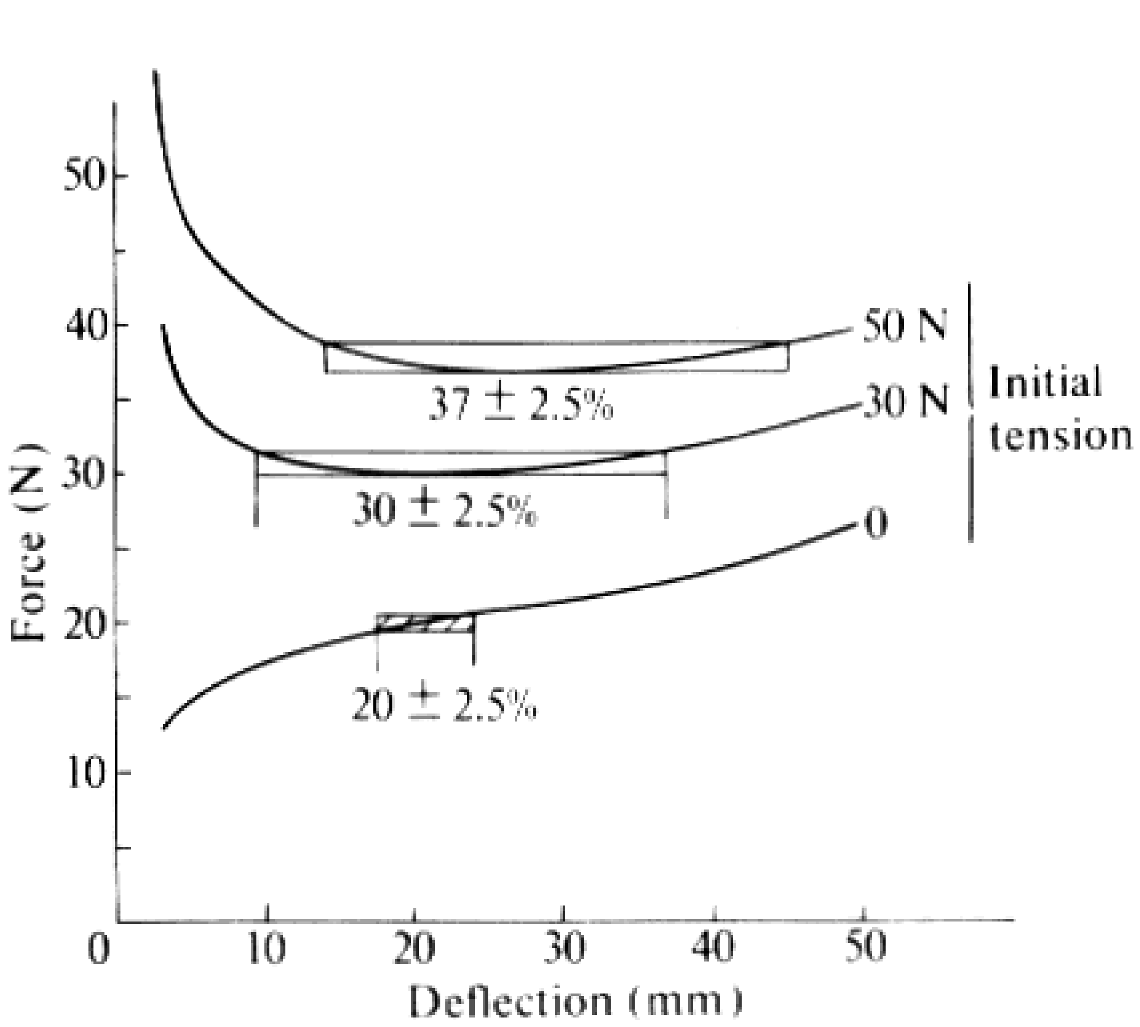

Fig. 13 Effect of varying initial tension in the body on the spring characteristic from zero to 30 N to 50 N

†As both trigonometrical functions and angles appear in the spring formulae, it is advisable to measure α and θ in the same units and in this case radian measure is used.

In the spring without initial tension the characteristic is similar to that of a helical compression spring, except that the first third shows a decreasing rate passing into a very narrow band (approximately 12 per cent) where the load rises by only 5 per cent.

The inclusion of 30 N initial tension raises the force by approximately 10 N and gives a characteristic that is concave upwards with a zone of approximately one half where the variation is less than ± 2.5 per cent.

Raising the initial tension to 50 N raises the force by a further 10 N and increases the zone to approximately two thirds of the total while shifting it bodily towards the end of the movement. There is, however, a limit to the amount of initial tension that can usefully be wound into a spring, which in this instance is closely approached by 50 N.

Lever arm length

The effect of increasing the length of the lever arm is threefold: (1) the range of the spring is increased almost in direct proportion, (2) the level of the zone of constant force is considerably lowered, and (3) the characteristic becomes markedly less concave. As will be seen later the bending stress in the arms becomes greater as their length increases until a point is reached where the spring is more likely to fail in bend than in torsion in the body of the spring. Load–deflection curves for the standard spring with lever arm lengths of 10 mm, 20 mm, and 40 mm are shown in Fig. 14.

Spring diameter

Examination of eqns (8) and (9), the two formulae from which the load–deflection characteristic of the coiled beam is derived, shows (1) that the relative deflection of the two arms is independent of the diameter of the helical portion of the spring, and (2) that the denominator of the force equation is diameter independent, whereas the numerator contains two terms, of which the first, $\theta G d^4/8nD$, is inversely proportional to the diameter while the second, $p_i D/2$, is directly proportional to it. This means that wherever the characteristic curves of two 'standard' springs differing only in spring diameter are plotted, then at some point those curves cross. To illustrate this the characteristic curves of 'standard springs' with mean diameters 14 mm, 15 mm, 16 mm, and 17 mm are shown on an exaggerated vertical scale in Fig. 15. It will be noted that all four curves cross at approximately 8 mm deflection and 32 N force, and that thereafter the curves are of very similar shape but with increasing slopes as diameter decreases, until at 50 mm deflection the forces are 31.4 N, 32.3 N, 33.8 N, and 35.8 N, in descending order of diameter. It will thus be seen that whereas the length of the lever arms must be closely controlled during manufacture to ensure uniformity of performance in the product, little attention need be paid to controlling the diameter of the body of the spring.

Number of coils

Of the terms governing the load–deflection characteristic, three are influenced by a change in the number of coils in the body of the spring and in each of these an increase in the number results in a decrease in the strength. That part of the torque resulting from the flexing of the body and not from initial tension, $\theta G d^4/8nD$, decreases in direct proportion to the increase in number of coils. The term $(nd/2\theta)\{\cos(\theta/2) - \cos\theta\}$, being part of the effective leverage, is directly proportional to the number of coils, a reduction of force resulting from any increase in this number. The linear deflection also contains a term directly proportional to n, i.e. $nd\{1 - (\sin\theta/\theta)\}$, and any increase will again result in a weaker spring.

Fig. 16 shows the load–deflection curves for the 'standard' spring having 10, 15, 20, and 25 coils in the body. It will be noted that although force decreases with increasing number of coils, the decrease is not proportional, and as the number of coils becomes larger the effect of each additional coil becomes progressively less.

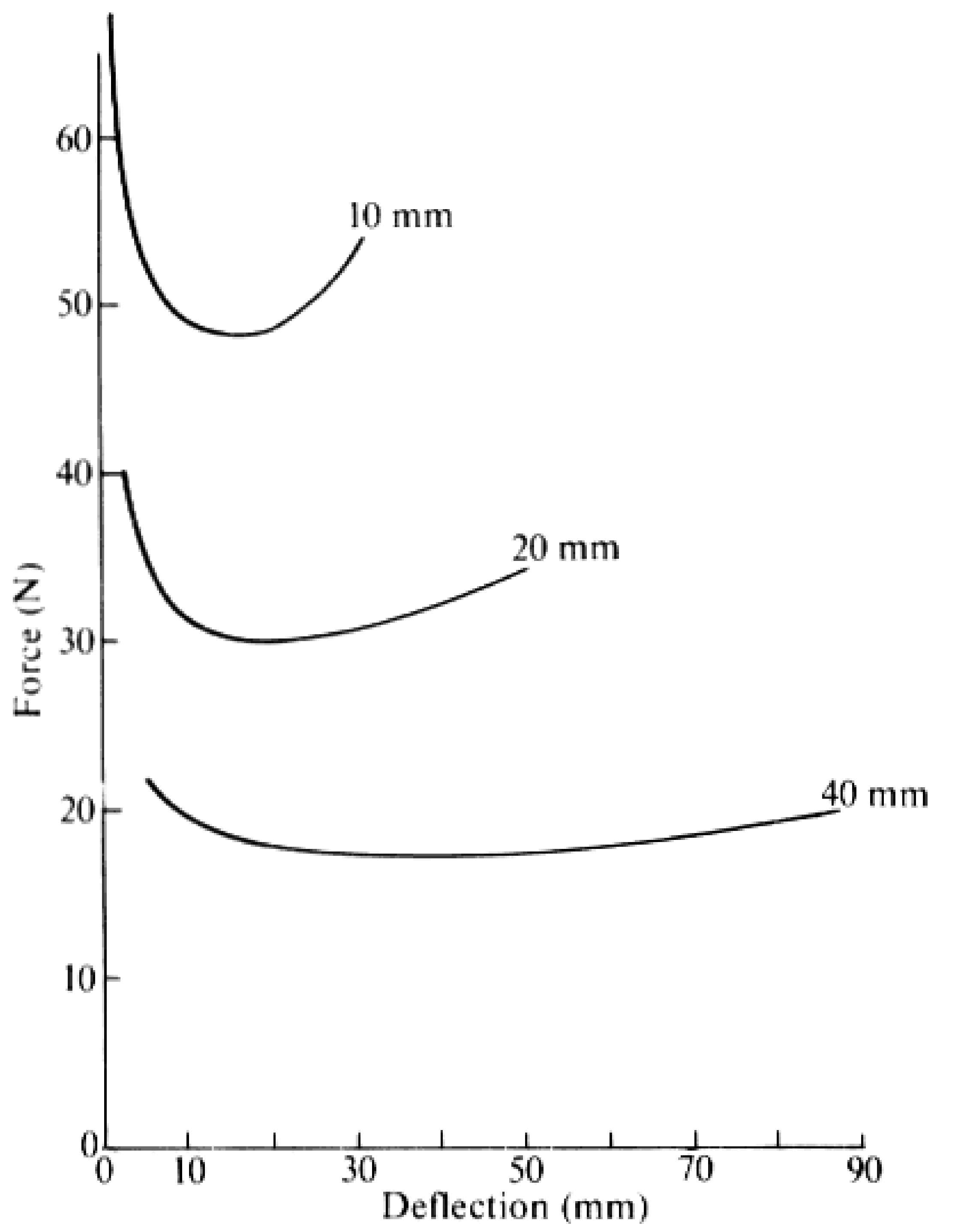

Fig. 14 Effect of varying arm length on the spring characteristic

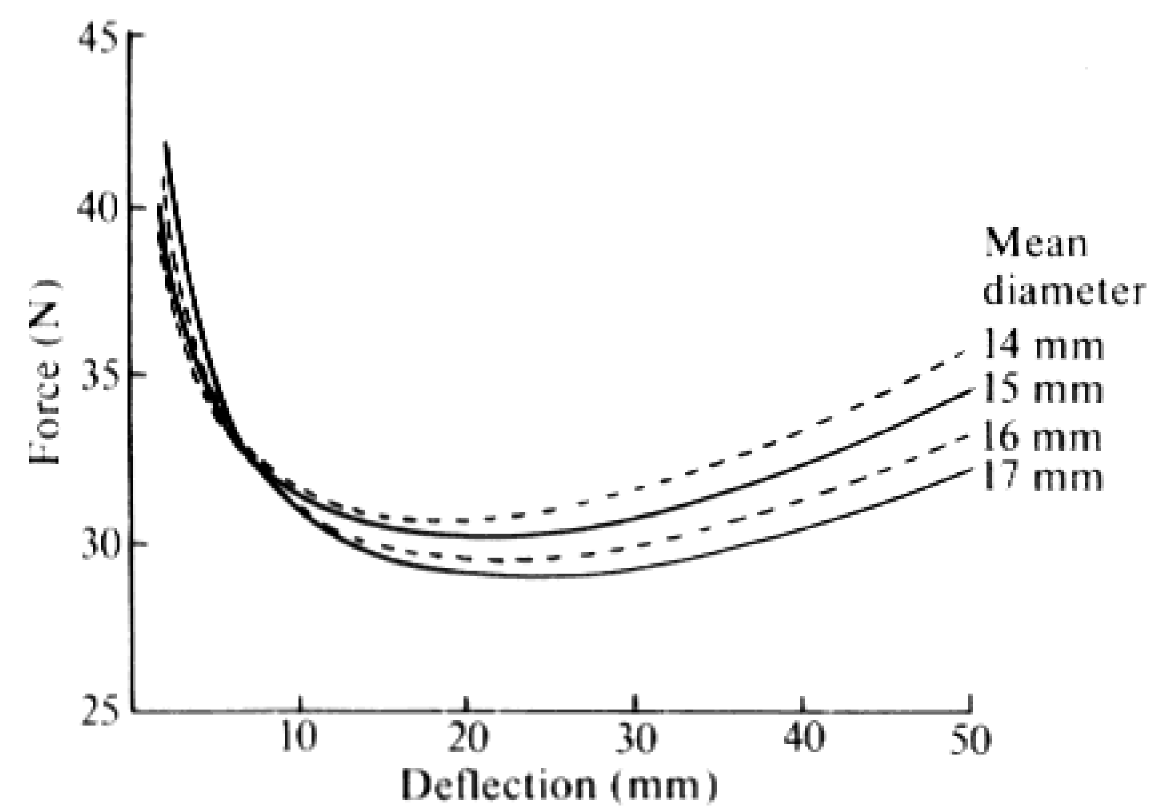

Fig. 15 Effect of varying mean diameter

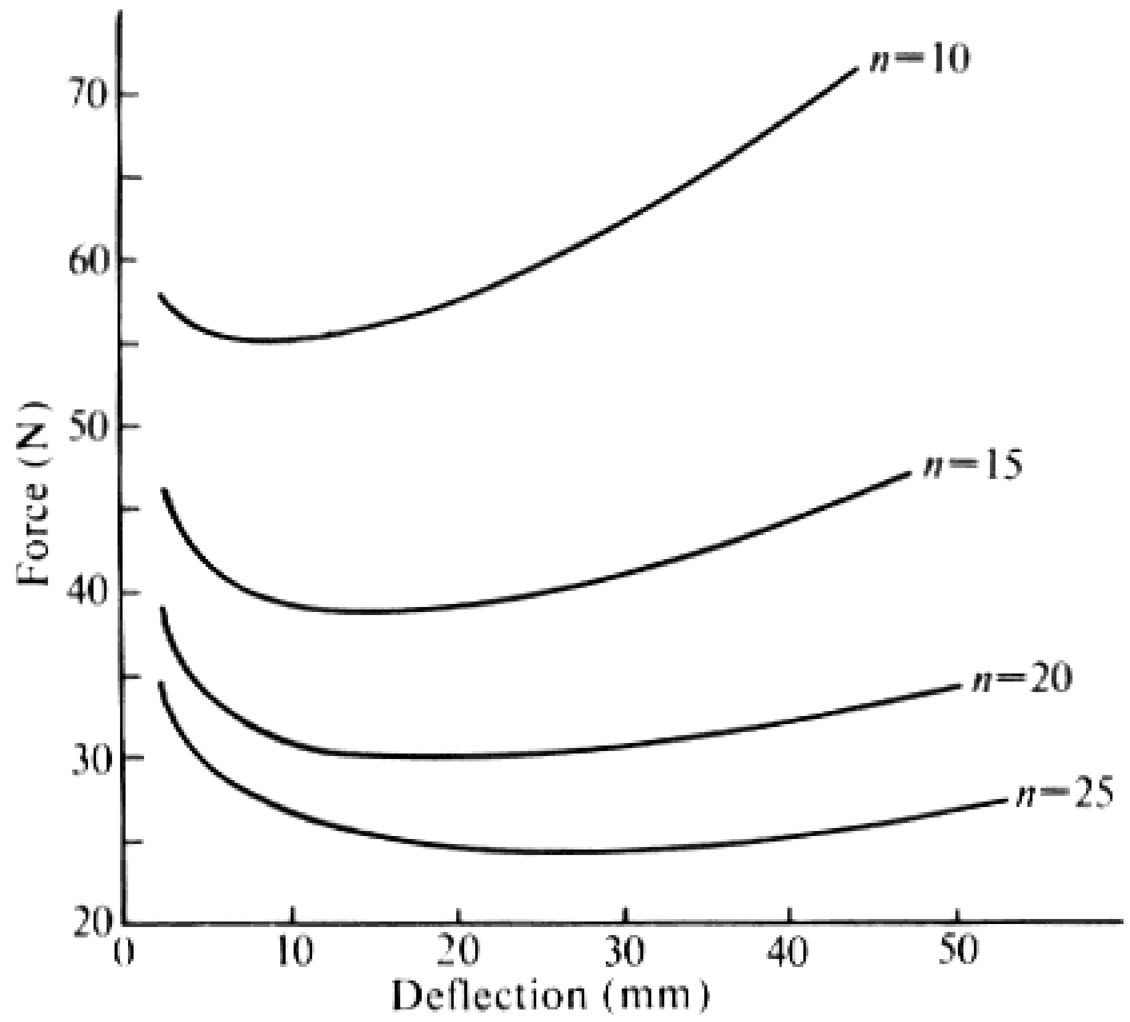

Fig. 16 Effect of varying the number of coils in the body

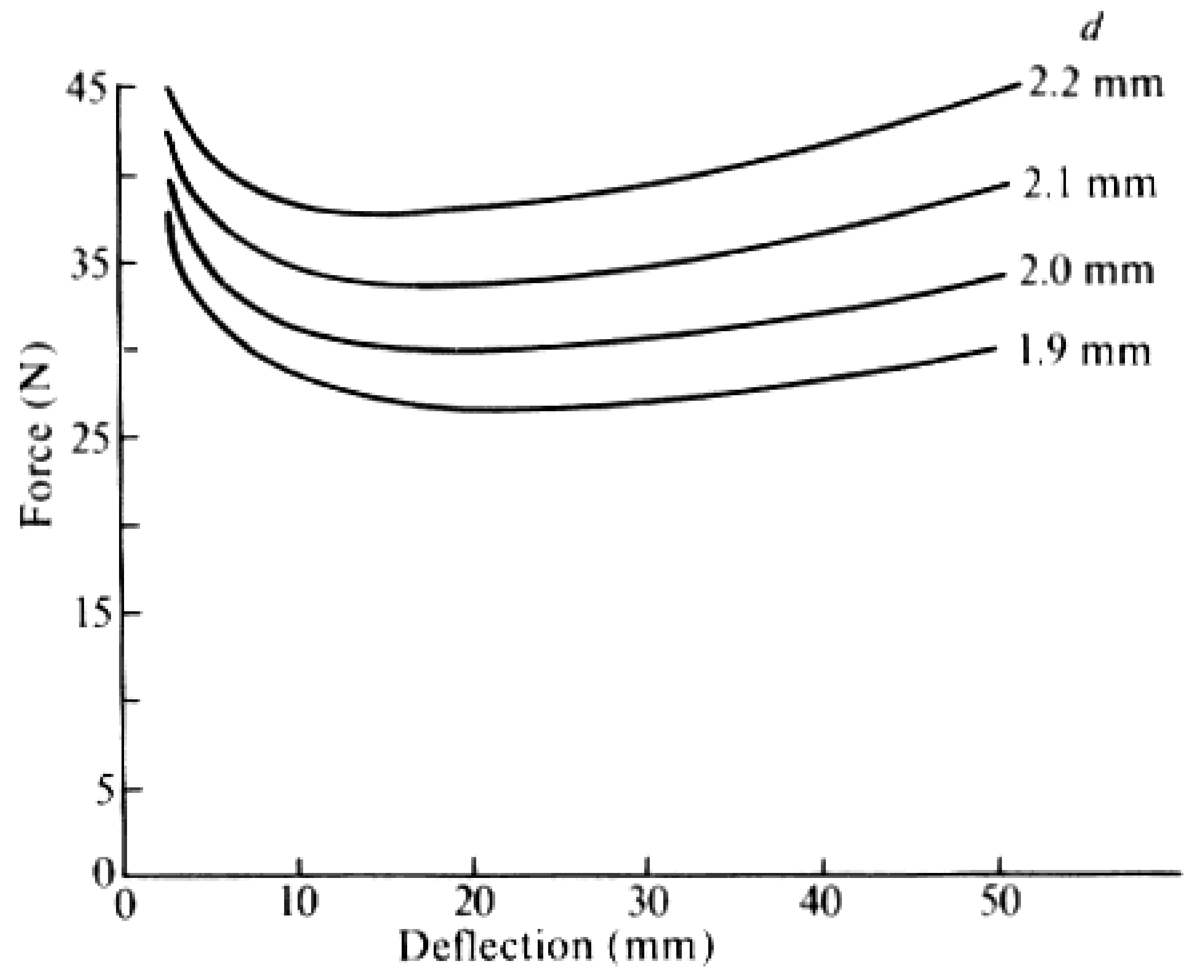

Fig. 18 Effect of wire diameter (*d*)

Angular deflection and starting angle

When the angular deflection of the arms, θ, reaches a value such that the radii from their instantaneous centres, through the eye centres, are parallel, the spring has generally reached the limit of its useful movement. It can be seen that after this position has been reached, that portion of the effective moment arm contributed by the spring arms decreases rapidly so that the value of that force increases in a similar manner. In the parallel position, of course, $\alpha + \theta = \pi/2$, and although it is possible to increase θ by making α negative this makes it necessary to preload the spring so that the application of load causes the spring arm eyes to approach each other in the required manner. One result of this method of assembly is that considerable moment is exerted while the effective lever arm is very short, and the implications of stress must be carefully considered. Fig. 17 shows the effect of changing the arm angle α from zero through 0.15 and 0.30 to 0.45 rad. It will be observed that although the effective deflection of the spring and the force exerted both decrease with increasing starting angle, the general shape of the main portion of the curve is not affected. If, however, the starting angle is below about 0.15 rad, then the first portion of the movement has a steep negative rate.

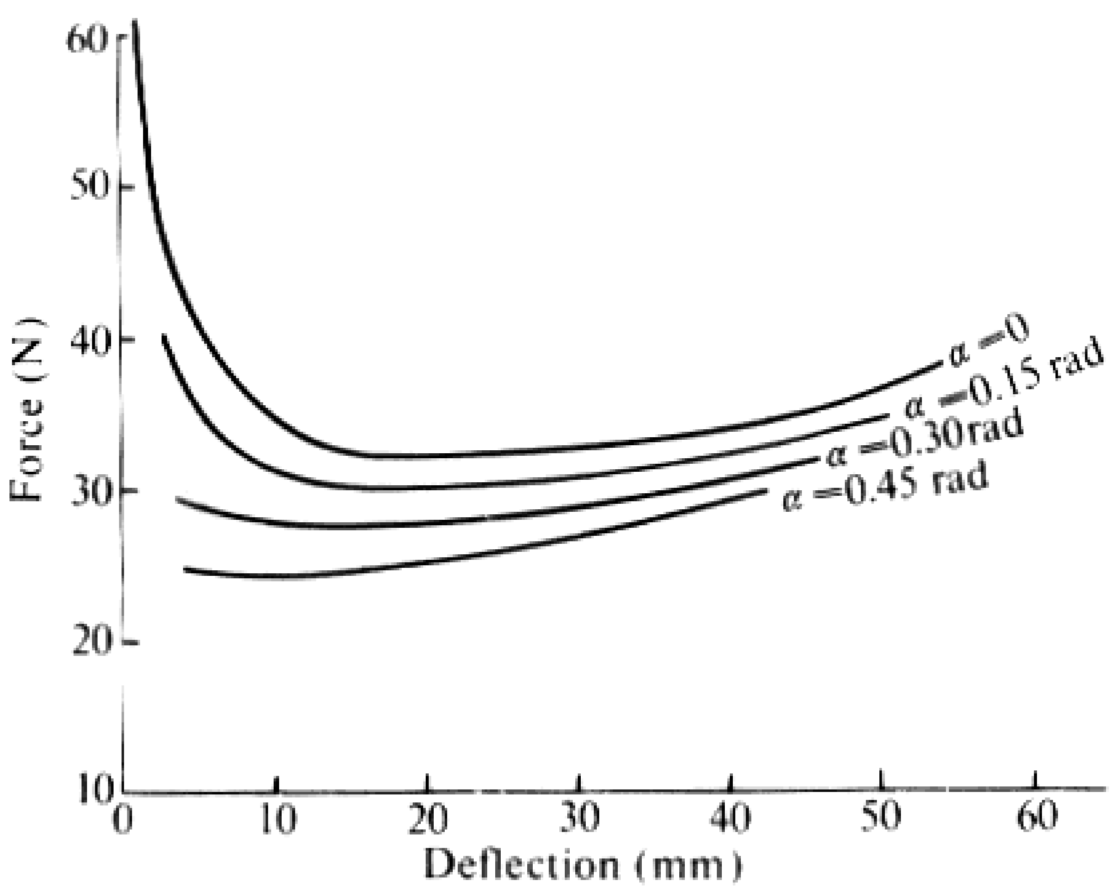

Fig. 17 Effect of starting angle on force and deflection

Wire diameter

Fig. 18 shows the effect of increasing the wire diameter while maintaining all the other spring characteristics. It will be seen that the force increases approximately with the third power of the wire diameter, whereas the available movement increases slightly owing to the increase in the body length of the spring. In addition, of course, the slight reduction in the ratio D/d of the spring would permit a higher level of useful initial tension so that the ultimate force would be still further increased.

Stress

The maximum shear stress occurs in that portion of the body which is furthest from the line of action of the force. If the arms are of equal length this is the middle coil and the stress decreases steadily from the centre to the ends. Using the notation on pp. 7–8 the maximum shear stress is given by

$$\tau_{max} = 16P(L + D)/\pi d^3, \qquad (10)$$

and for static or near-static conditions the same limits may be set as for extension springs made from the same material.

The maximum bending stress occurs at the points at which the arms leave the body of the spring and may initially be calculated as

$$f_{b_{max}} = 32\ Pl/\pi d^3. \qquad (11)$$

It is, however, necessary to allow for the combination of bending and torsional stresses at these points by applying a correctoin factor, K_c, similar to that applied to the eyes of helical extension springs. Opinions vary considerably as to the actual numerical value that the factor should assume, but in the author's experience,

$$K_c = (c + 8)/(c + 5), \qquad (12)$$

where $c = 2r/d$ (r being the smallest mean radius of curvature at the 'blend'), gives adequate protection. Thus for the standard spring specified on p. 8, if care is taken that at no point in the 'blend' does the radius fall below half that of the body of the spring, then

$d = 2$ mm, $D = 15$ mm, $r = 3.7$ mm (minimum), whence the maximum correction factor $\{(7/2) + 8\} \div \{(7/2) + 5\} = 1.35$.

Care must be taken to see that both modes of stress lie within the capacity of the material.

Design guidelines

When designing a spring to given performance and space requirements the following guidelines should be borne in mind:

(1) the available moment is approximately twice the arm length plus one-third of the body length, i.e. $2l + (nd/3)$;

(2) the space needed to one side of the line of action of the force is approximately the length of the arm plus one third of the body length, i.e. $l + (nd/3)$;

(3) the angular movement available is approximately $\pi/2$ minus the starting angle, i.e. $(\pi/2) - \alpha$;

(4) a 'flat' characteristic with low force is achieved by using long arms and high initial tension, and

(5) a 'flat' characteristic with higher force is achieved by using shorter arms with high initial tension although some of the 'flatness' will be lost. Alternatively the wire size may be increased.

Worked example

It is desired to maintain a force of 6.4–6.9 N over a distance of approximately 50 mm. The spring-eye bearing centres should be approximately 30 mm apart at full compression and the available space to one side of the line of action of the force is restricted to a maximum of 50 mm.

This specification calls for a characteristic with as great a horizontal section as possible. The best that can be achieved with an orthodox spring is about two thirds of the total, so that the total travel must be approximately $50 \times (3/2) = 75$ mm. Thus:

(1) $2l + (nd/3) = 75$, and

(2) $l + (nd/3) = 45$ (this may if necessary be varied up to 50 mm). From (1) and (2) $l = 30$ mm and $nd = 45$, and at this stage take $\alpha = 0.15$ rad as a suitable starting point.

(3) An approximate wire size, d, may be derived from eqn (10)—

$$\tau_{\max} = \frac{16P\{L + (nd/3)\}}{\pi d^3},$$

and if the approximate uncorrected torsional stress is taken as 440 N/mm^2, then

$$d^3 = \frac{16 \times 6.65 \times 45}{440 \times \pi} = 3.46,$$

whence $d = 1.51$ mm (1.5 mm) and $n = 30$.

In order to ensure that the stress-correction factor at the 'blend' between body and arm, K_c, is not too high, start with a spring ratio, D/d, of just over 8 and be prepared to vary this if necessary.

Work done by The Spring Research and Manufacturers' Association indicates that a reasonable initial tension level for a spring of these dimensions would be 15 N.

The design is therefore 30 coils of 1.5 mm wire, arm length of 30 mm, starting angle of 0.15 rad, initial tension 15 N, and mean diameter of approximately 13 mm. Check calculations give the results listed in Table 1. Thus the maximum torsional stress in the body of the spring is given by

$$\tau_{\max} = \frac{16 \times 7.1 \times (39.64 + 13)}{\pi \times 1.5^3} = 560 \text{ N/mm}^2,$$

and the maximum combined stress in an arm using the correction factor, K_c, as specified is given by r.h.s. eqn (11) × r.h.s. eqn (12) as follows:

$$f_{b,t_{\max}} = \frac{32 \times 7.1 \times 30}{\pi \times 1.5^3} \times \frac{4.35 + 8}{4.35 + 5} = 843 \text{ N/mm}^2.$$

Both of these values are within the static capabilities of any good-quality patented spring steel wire.

The load–deflection characteristic of the spring is shown in Fig. 19.

Table 1. Reference data for worked example

θ (rad)	Moment of force (N mm)	Effective lever (mm)	Force (N)	δ (mm)
0.2	120.8	11.96	10.1	3.3
0.4	144.4	19.00	7.6	9.4
0.6	174.4	25.27	6.9	18.1
0.8	199.5	30.69	6.5	29.1
1.0	223.8	34.97	6.4	41.9
1.2	250.3	37.93	6.6	56.2
1.4	276.9	39.55	7.0	71.1
$(\pi/2) - \alpha$	281.4	39.64	7.1	72.2

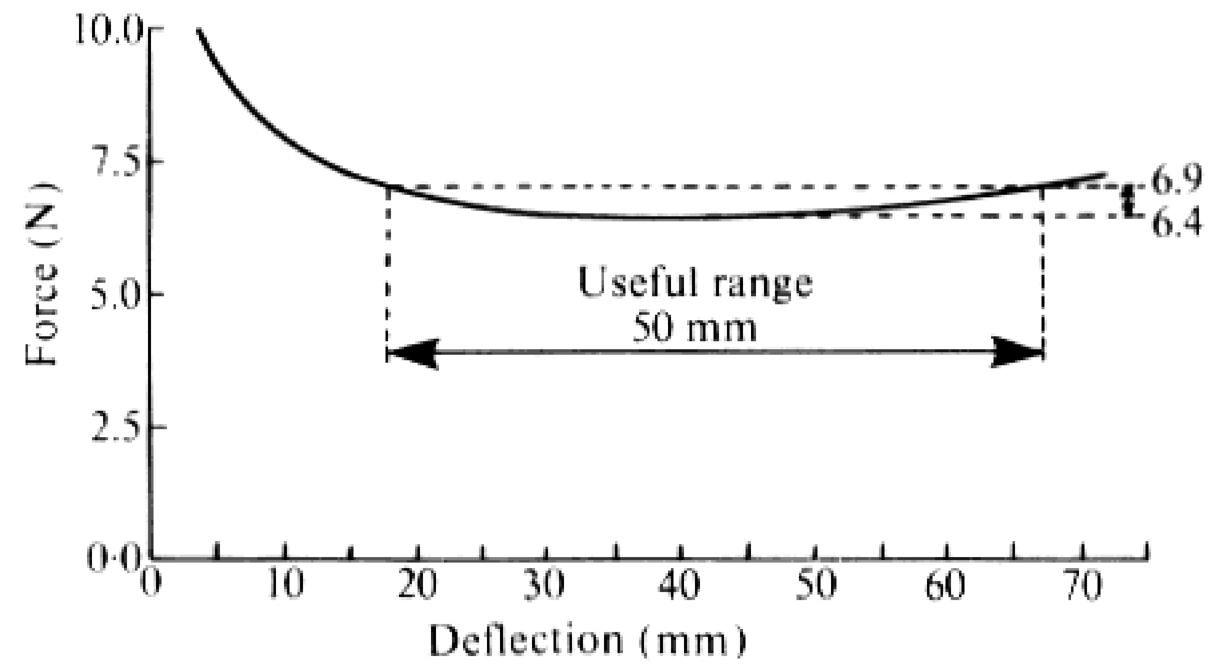

Fig. 19 Solution to worked example

Materials and applications

It has been shown that in order to produce the desired zero-rated characteristic as well as a high level of force, it is necessary to use a high level of initial tension in the body of the spring. It is therefore obvious that the material used must be 'spring hard' before coiling and must not require any high-temperature heat treatment after cooling. Patented cold-drawn steel spring wire (to BS 1408), rust-, acid-, and heat-resistant steel spring wire (to BS 2056), oil-hardened and tempered steel wire for springs (to BS 2803), Ni-Span C and D,† and the multitude of 'spring-temper' copper alloys are all suitable materials, each of which offers its own particular advantages to the designer.

By far the greatest use of springs of this type lies in the electrical field, where it is necessary to compensate for the loss in length of carbon brushes resulting from wear without the loss in force that is associated with the orthodox compression spring. As

† Henry Wiggin & Co. Ltd.

the force remains constant within ± 5 per cent over the selected travel, the problem of arcing and resulting commutator damage is greatly reduced. It is of course possible to specify closer limits than ± 5 per cent but there is always some penalty to pay such as reduction in range, lower level of force, and greater space requirement, and the designer must examine the alternatives given in Figs 13–18 and decide for himself what should be sacrificed in favour of uniformity of force. Levels of ± 3.5 per cent are obtainable but should not be specified unless absolutely necessary as they can involve secondary operations to ensure that an acceptable percentage of the product is within tolerance.

Leaf springs

Cantilevers

The simplest form of leaf spring, known as the cantilever, is of course the single leaf of constant rectangular cross-section clamped firmly at one end and loaded at the other. Unfortunately, although simple and inexpensive to produce, it utilizes the material in a very inefficient manner as will become apparent from the following examination of the formulae governing a spring of this type.

Fig. 20 shows a single leaf of length l and rectangular cross-section bt, carrying load P at its free end.

The deflection at a distance x from the fixed end is given by

$$\delta_x = \frac{l^3 P}{6EZ} \times \frac{x^2}{l^2}\left(3 - \frac{x}{l}\right),$$

where E is Young's modulus for the material, and Z is the modulus of section (in this example $t^3b/12$) so that:

$$\delta_x = \frac{2l^3 P}{Ebt^3} \times \frac{x^2}{l^2}\left(3 - \frac{x}{l}\right).$$

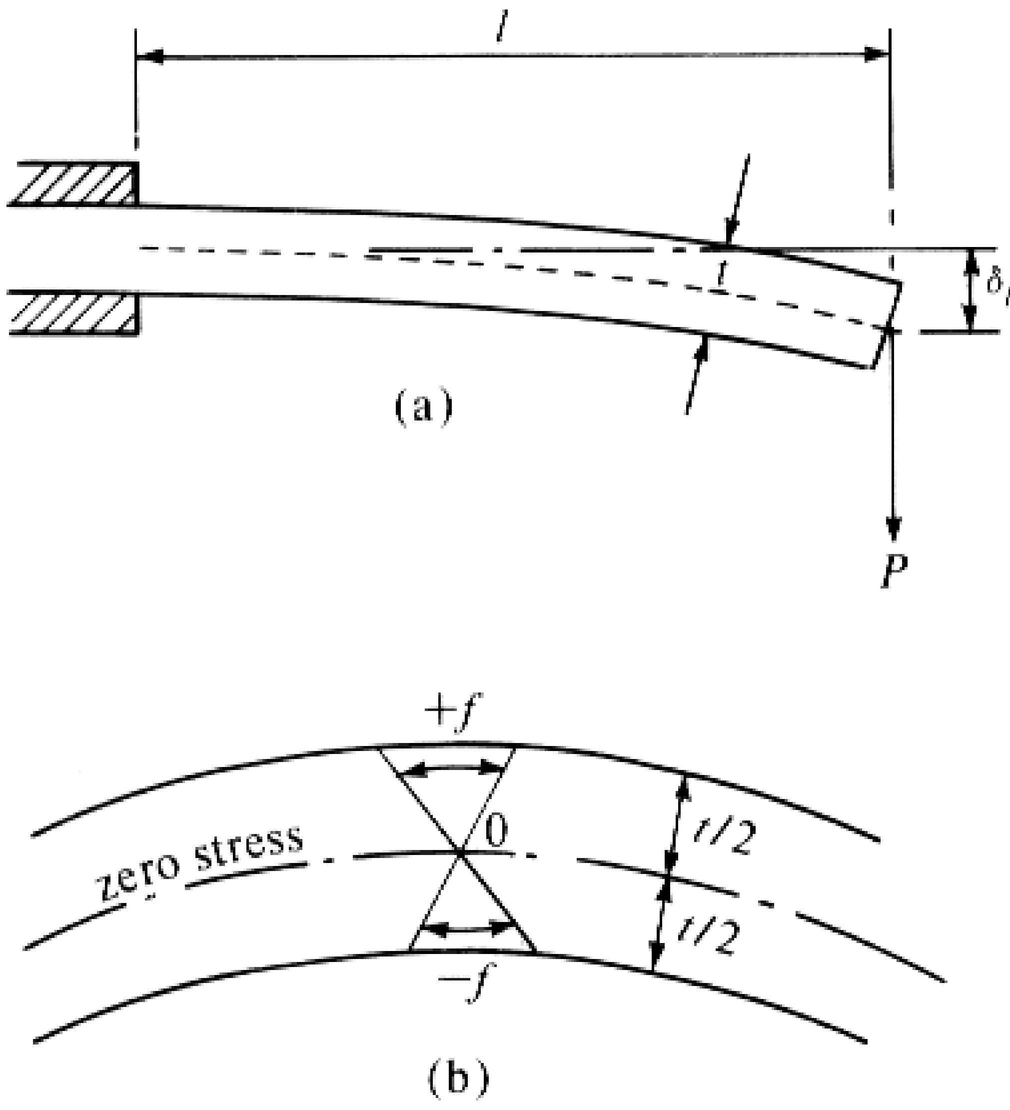

Fig. 20 Cantilever spring: (a) single leaf of rectangular cross-section, (b) stress distribution across section

If zero and l are substituted in turn for x, the deflection at the fixed end is zero (as would be expected) and at the free end

$$\delta_l = 4l^3P/Ebt^3. \qquad (13)$$

The stress at a distance x from the fixed end is given by

$$f_x = \frac{tPl}{2Z}\left(1 - \frac{x}{l}\right) = \frac{6Pl}{bt^2}\left(1 - \frac{x}{l}\right)$$

Substituting zero and l for x:

$$f_0 = 6Pl/bt^2 \qquad (14)$$

and

$$f_l = 0,$$

where f_0 is stress at fixed end and f_l is stress at free end.

It can thus be seen that the stress tapers off uniformly from a maximum to zero and that there appears to be only a single cross-section where the material is efficiently used. This would be bad enough, but examination of any cross-section shows (see Fig. 20) that the fibres on the convex side are stressed in tension and those on the concave side in compression, and that the stress varies from a maximum positive (tensile) stress at the convex surface, through zero at 0, to a maximum negative (compressive) stress at the concave surface, so that there is a plane parallel to both faces and passing through the centre of gravity of each cross-section which is completely unstressed. Fortunately there are many applications where manufacturing cost is the overriding criterion and this, coupled with the ease with which such a spring may be mounted, leads to its great popularity. Fig. 21 shows the application of a simple single-leaf spring in a device for measuring the profile of the ground end of a helical spring relative to its major axis. A self-centering four-jaw chuck is provided with a series of twelve (or any other preferred number) equally spaced hemispherical indents round its circumference and is mounted rotationally on a suitable face plate. A single-leaf spring having a round-headed bolt to match the indents fixed to its free end is clamped in the position shown by the dotted line before the chuck is mounted, and then 'sprung' aside while the bearing pin is dropped into its bearing. The spring is then released and the chuck rotated until the bolt head mates with one of the indents. The spring to be tested is mounted in the chuck and a clock gauge mounted above and in contact with the ground face. By rotating the chuck until each indent has successively mated with the spring a series of twelve equally spaced readings may be obtained.

In order to compare other types of leaf spring with this simple form, consider a spring in which $b = 25$ mm, $t = 6$ mm, and $l = 300$ mm of a material with a modulus of 207 000 N/mm², stressed to 1200 N/mm². The unit of comparison is taken as K, where K = *force* × *deflection*/*volume*. Since f =

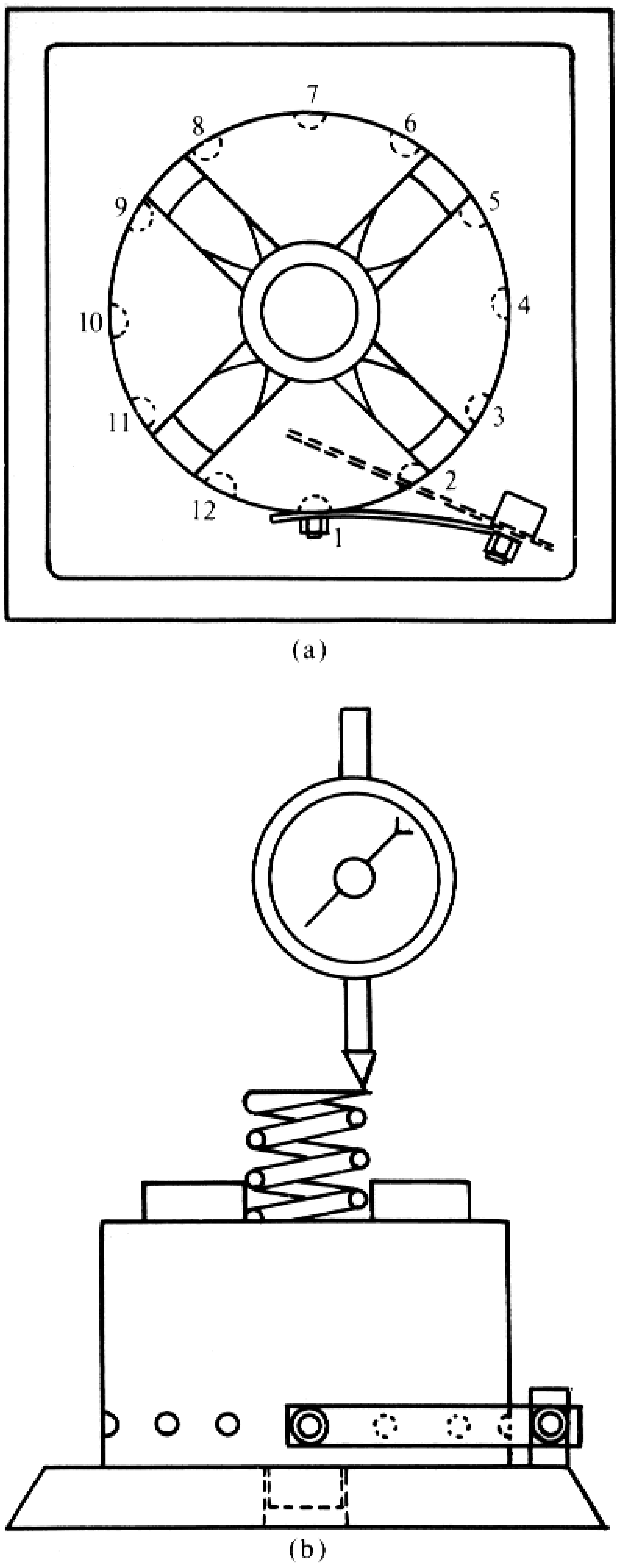

Fig. 21 Spring- and profile-measuring device using single cantilever spring: (a) plan, (b) front elevation

1200N/mm², then from eqn (14):

$$P = \frac{1200 \times 25 \times 6^2}{6 \times 300} = 600 \text{ N}$$

From eqn (13);

$$\delta_l = \frac{4 \times 300^3 \times 600}{207\,000 \times 25 \times 6^3} = 58 \text{ mm.}$$

Also, volume,

$$V = 300 \times 6 \times 25 = 45\,000 \text{ mm}^3.$$

Thus,

$$K = 600 \times 58/45\,000 = 0.77.$$

Because the stress varies in a linear manner along the length of the leaf and is also inversely proportional to its width, it would appear in theory that a state of constant stress can be achieved by tapering the width from full to zero while maintaining a constant thickness; thus in plan the leaf is an isosceles triangle. Under these conditions the deflection is given by

$$\delta_l = 6l^3P/Ebt^3,$$

where b is the width at the clamped end. The leaf under load assumes the form of a circular arc and the stress remains constant along its length as $f = 6Pl/t^2b$, while $V = blt/2$.

Applying the load and stress conditions laid down for the rectangular leaf, $P = 600$ N, $\delta_l = 87$ mm, and $V = 22\,500$ mm³, whence

$$K = 600 \times 87/22\,500 = 2.32.$$

which is three times the standard value.

As previously stated, this can only be achieved in theory since no force can be applied at right angles to the plane of the leaf through a point having neither width nor breadth in that plane. However, so great is the possible saving in weight achieved by this method that where this is the main consideration a little of the advantage may be forfeited by providing excess metal at the pointed end as shown in Fig. 22(a).

The third type of single leaf, the truncated triangle or trapezoidal spring, is shown in Fig. 22(b). Here b_0 is the width of the leaf at the fixed end and b_l the width at the point of application of load. Note that this is not at the physical extremity of the leaf but at the limit of the stressed portion, the excess being

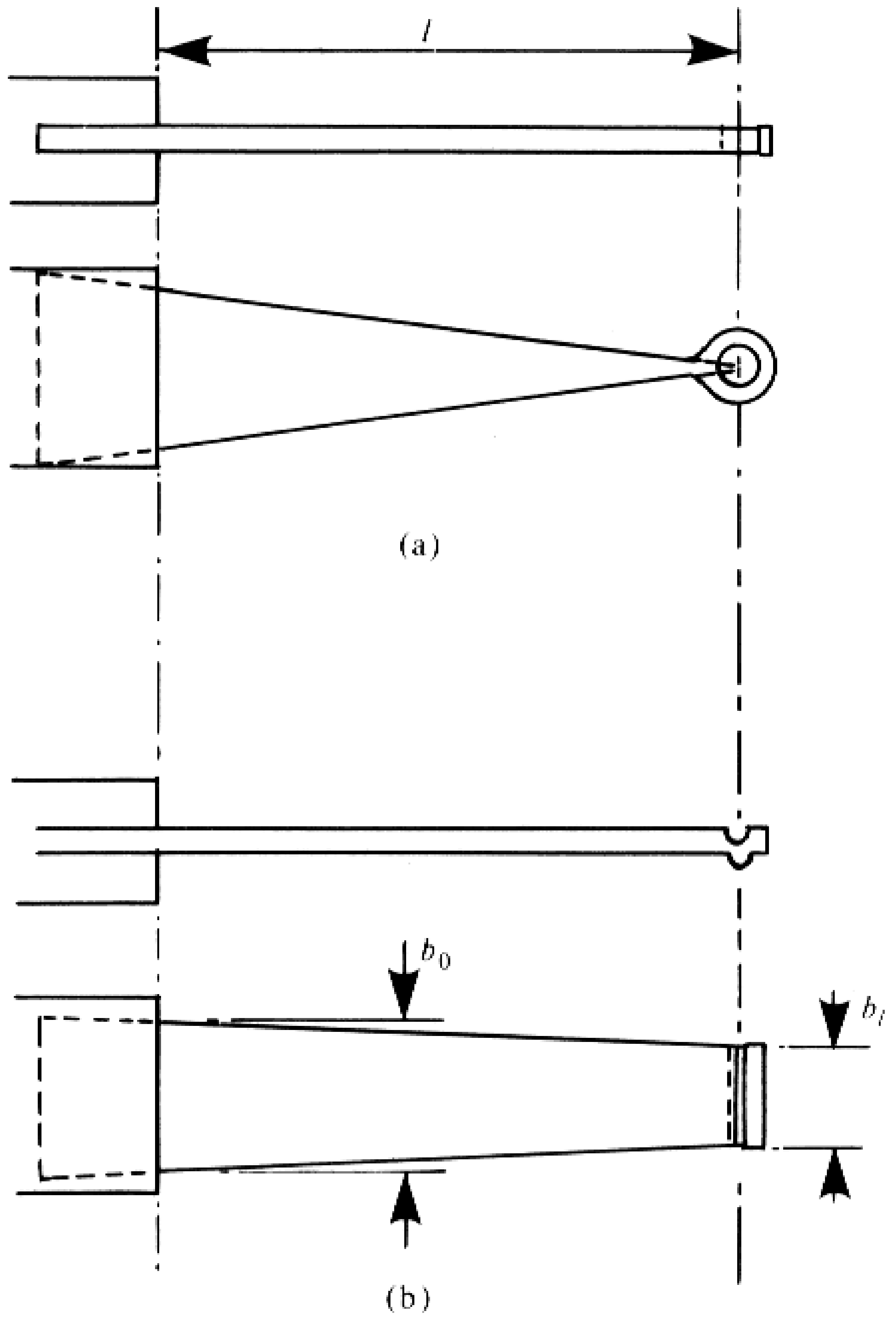

Fig. 22 Constant stress cantilevers

necessary to provide anchorage for the load. This extra length for anchorage is also a necessary part of the rectangular leaf.

The deflection under load, when compared directly with the standard leaf, has to be modified by a constant, M, which is a function of the ratio of b_0 and b_l, as shown in Fig. 23. A simplified solution to this curve is given by

$$M = 3b_0/(2b_0 + b_l),$$

from which it will be observed that when $b_0 = b_l$ then $M = 1$, and when $b_l = 0$ then $M = 1.5$. This means that when the curve passes through the correct point at both extremes the simplification results in an optimistic error of just under 5 per cent in the middle range. In Fig. 23 the approximation is shown for comparison purposes as a broken line beside the true curve.

For this type of spring the factor K lies between the standard value of 0.77 and a maximum of about 2.0 as the taper increases.

A second method of achieving uniformity of stress along the length of the leaf, and thus greater efficiency, consists of tapering the thickness to comply with the form of a parabola, where

$$t_x = t_0\sqrt{\{1 - (x/l)\}},$$

$\delta_l = 8l^3P/bt^3E$, and $V = 2btl/3$. Thus

$$K = 600 \times 116/30\,000 = 2.32,$$

which is the same value as for the triangular spring.

To summarize: if ease of production and assembly are the overriding considerations, then a rectangular leaf of uniform thickness should be used. If however the saving of material (because of either expense or weight) is of major importance, then a trapezoidal leaf of uniform section or a rectangular leaf of parabolic form would be more suitable.

One other form in which the rectangular leaf of uniform thickness finds use is as a measurer of precise time intervals. The natural frequency of such a leaf is given by

$$\nu = 1.015t(Eg/\rho)^{\frac{1}{2}}/l^2,$$

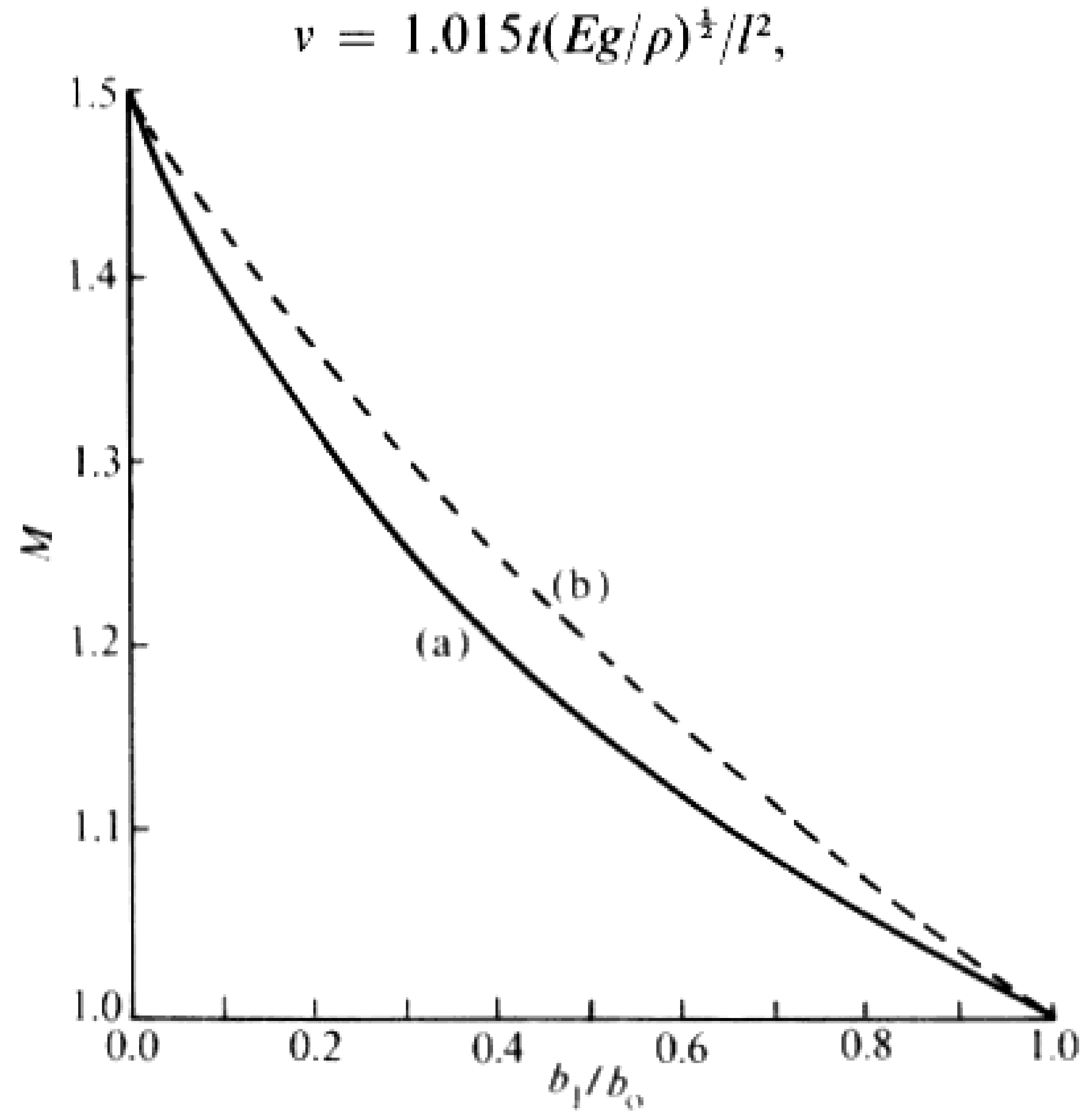

Fig. 23 Modification constant, M, for trapezoidal leaf: (a) true curve, (b) $M \simeq 3b_0/(2b_0 + b_l)$

where g is the acceleration due to gravity and ρ is the density of the material, both measured in the same system of units. Thus if E and ρ are both known ν may be calculated, whereas measurement of ν leads to easy calculation of E for new or unknown materials.

Provided that the deflection is small relative to the length of bar, then the deflection and stress formulae hold good whether or not the initial condition of the bar is straight.

Cantilever spring assemblies of increasing rate

An even more efficient use of the material may be made by providing the cantilever with a curved stop which allows each portion to reach, but not to exceed, the design stress. This is illustrated in Fig. 24(a),

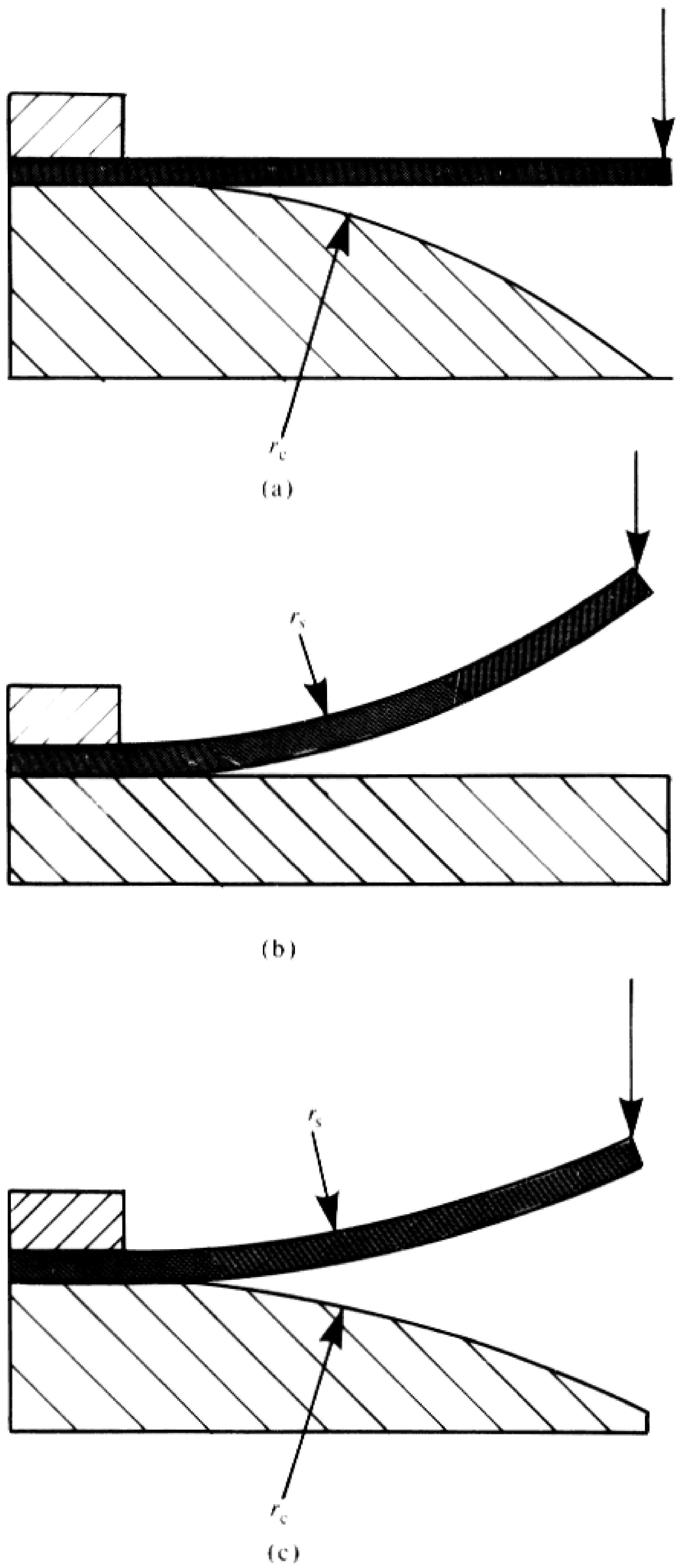

Fig. 24 Cantilever assemblies for increasing rate: (a) $r_s = \infty$, $1/r_c = 2f_D/Et$; (b) $r_c = \infty$, $1/r_s = 2f_D/Et$; (c) $1/r_c + 1/r_s = 2f_D/Et$

where the stop is in the form of a circular arc of radius r_c. The same effect may be achieved by using a flat stop and a cantilever spring that is initially formed to a circular arc of radius r_s (see Fig. 24(b)). A further modification that may be dictated by the exigencies of space uses an initially curved spring (r_s) and a curved stop (r_c) as shown in Fig. 24(c). In all three instances the radii used are governed by the formula:

$$\frac{1}{r_c} + \frac{1}{r_s} = \frac{2f_D}{Et},$$

where f_D is the design stress, E is the modulus of elasticity, and t is the material thickness. Under these conditions the spring deflects as a free cantilever until the design stress is reached, at which point contact is made with the stop, and as deflection continues the active length progressively decreases, the rate increases, and the stress remains constant once it has reached the design limit. Fig. 25 shows a typical force–deflection curve where the ratio *deflection/safe deflection of the unsupported spring* is plotted against *force/maximum force of the unsupported spring.*

This principle, modified so that the stop is formed by a spring that is a mirror image of the one above it, is used in the repeating rifle magazine spring (see Fig. 26). Each pair of springs consists of a pair of double cantilevers connected by a circular portion of sufficient radius to facilitate forming without risk of cracking, in which a gap is left so that when the platform is depressed the gap closes at the design stress and allows the two springs to come into contact and roll on each other. When the magazine is full, the spring is closed into a compact rectangular block giving a very high energy–weight ratio.

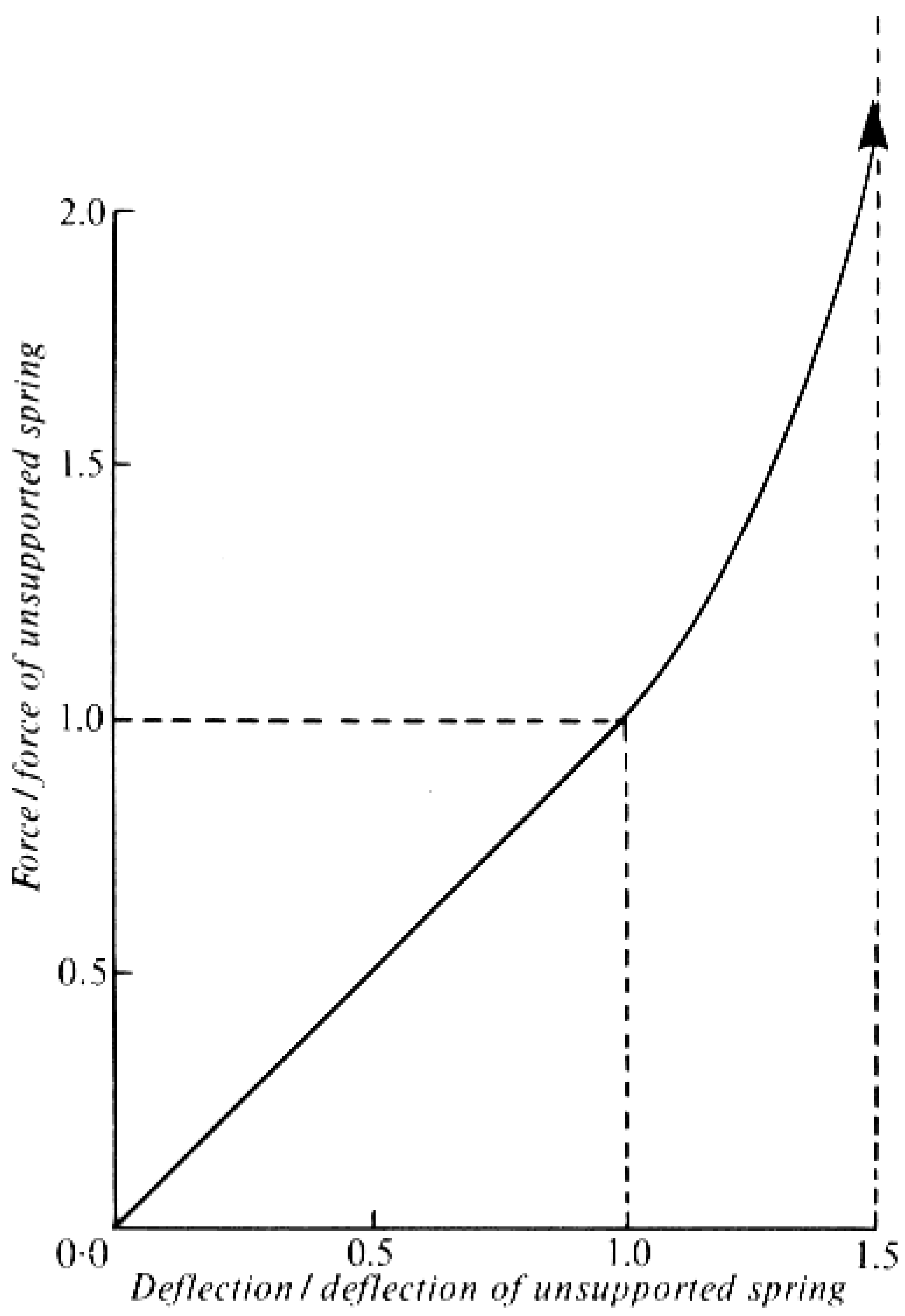

Fig. 25 Force–deflection diagram for increasing-rate cantilever

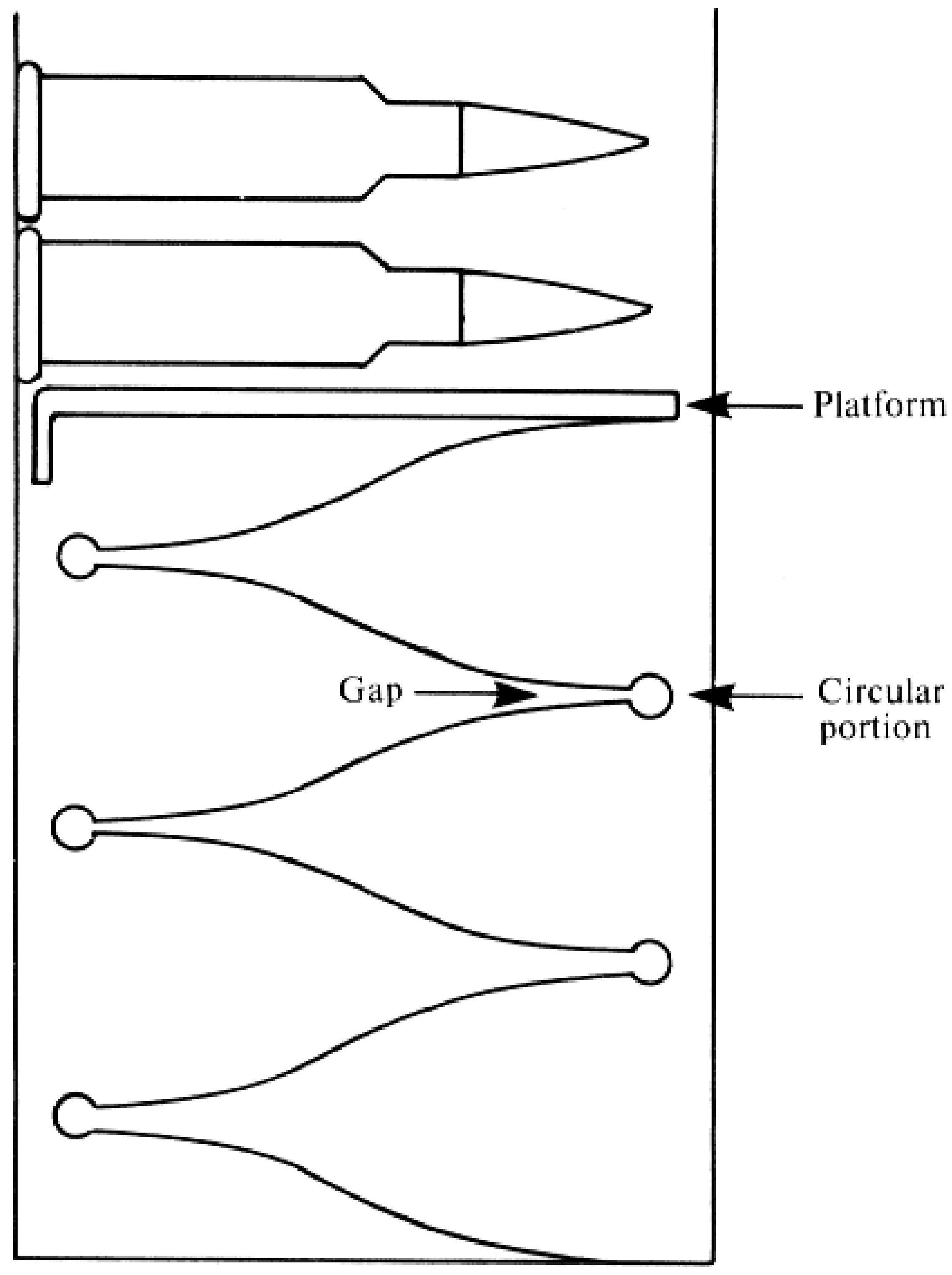

Fig. 26 Automatic rifle magazine spring

Double cantilever springs

If a single, straight, rectangular-section spring is placed on two supports in the same plane and loaded in the centre, the two halves behave as single cantilevers (Fig. 27(a)). If the distance between the supports is l and the load is P, each half may be considered to be a cantilever of length $l/2$ carrying a load $P/2$.

From eqn (13) the deflection at the centre is given by

$$\delta_l = 4 \times (P/2) \times (l/2)^3/Ebt^3 = \tfrac{1}{4}l^3P/Ebt^3,$$

which is one-sixteenth that of a cantilever of the same

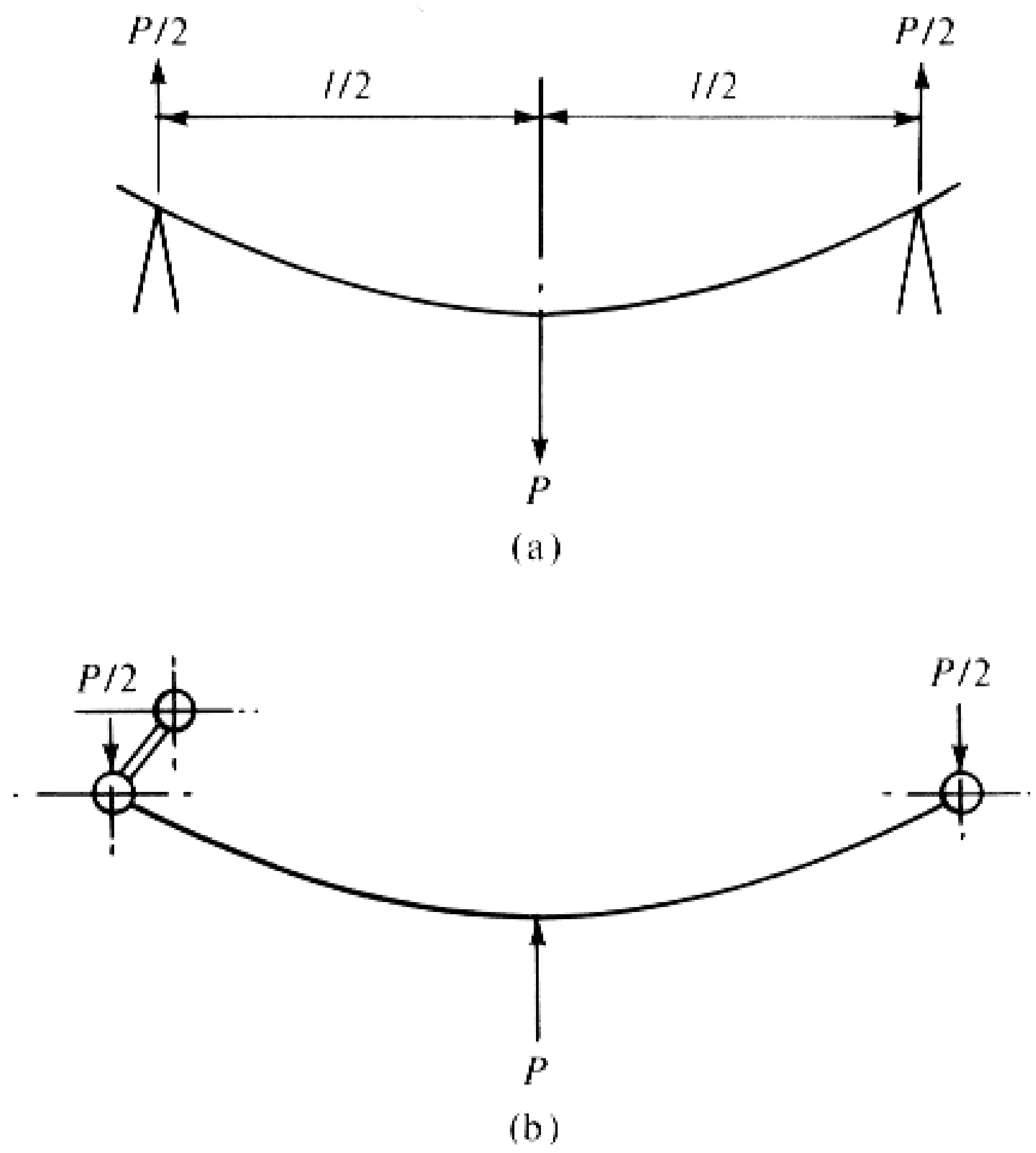

Fig. 27 Double cantilever springs: (a) beam spring, (b) carriage spring

length. From eqn (14) the stress at the centre is given by

$$f_0 = 6(\tfrac{1}{2}P)(\tfrac{1}{2}l)/bt^2 = 3Pl/2bt^2,$$

which is one-quarter that of the single cantilever.

This type of spring is not commonly used in this form since some system of lateral guidance would be necessary to maintain the spring in position. However, with the leaf pre-formed into a slight curve and the load applied at both ends it forms the basis of the 'cart' or 'carriage' spring (see Fig. 27(b)).

Since the loading of such a spring reduces either its height or its camber, the ends move away from each other so that rigid fixing is impossible and both ends must be provided with some form of bearing to allow it to rotate, while one end must be suspended on a swinging link to accommodate the change in overall length. As long as the free camber is not too great the formulae derived from eqns (1) and (2) may be used, although it will be obvious that as the lever arm increases the stress will increase in proportion and some small factor of safety should be used.

If a single leaf spring is clamped at both ends and those ends constrained to move in guides in a direction at right angles to the major axis of the spring as shown in Fig. 28, the assembly may be considered as two cantilevers each of length $l/2$ and the deflection and stress may be calculated exactly as before.

This type of spring is commonly used on vibrating screens actuated by an eccentric for the purpose of transferring loose material, such as root crops and stones, from one point to another and at the same time 'sizing' it by means of graduated apertures in the screen. This, of course, subjects the springs to a compressive load that is additional to the bending moment imposed by the horizontal displacement of the moving end (see Fig. 29). Under these circumstances, if P is the load supported by a single spring and Q is the displacing force, then the total stress, f, is given by

$$f = -P/bt \pm \left(\frac{6Q \tan wl}{wbt^2}\right),$$

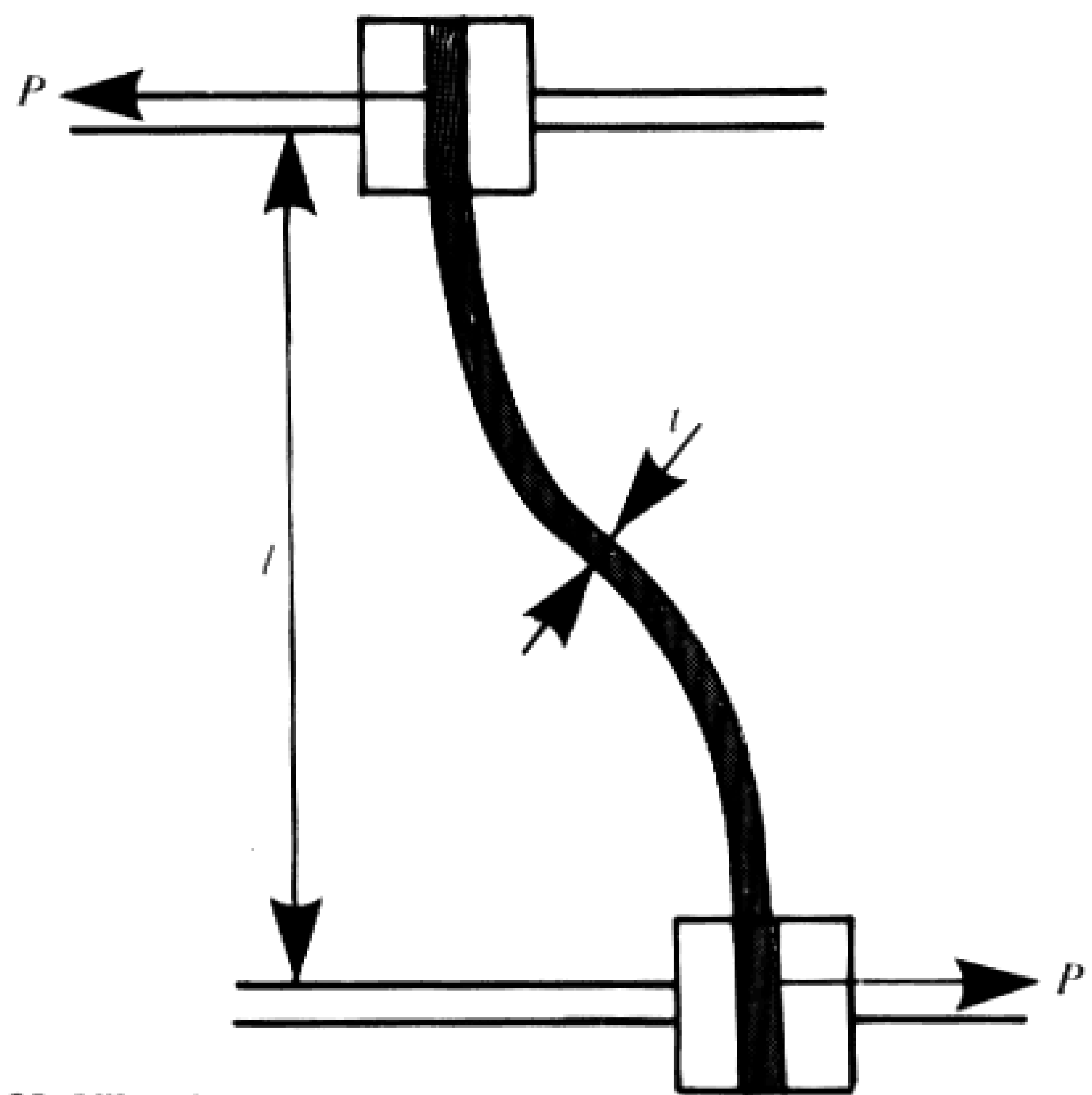

Fig. 28 Vibrating screen spring

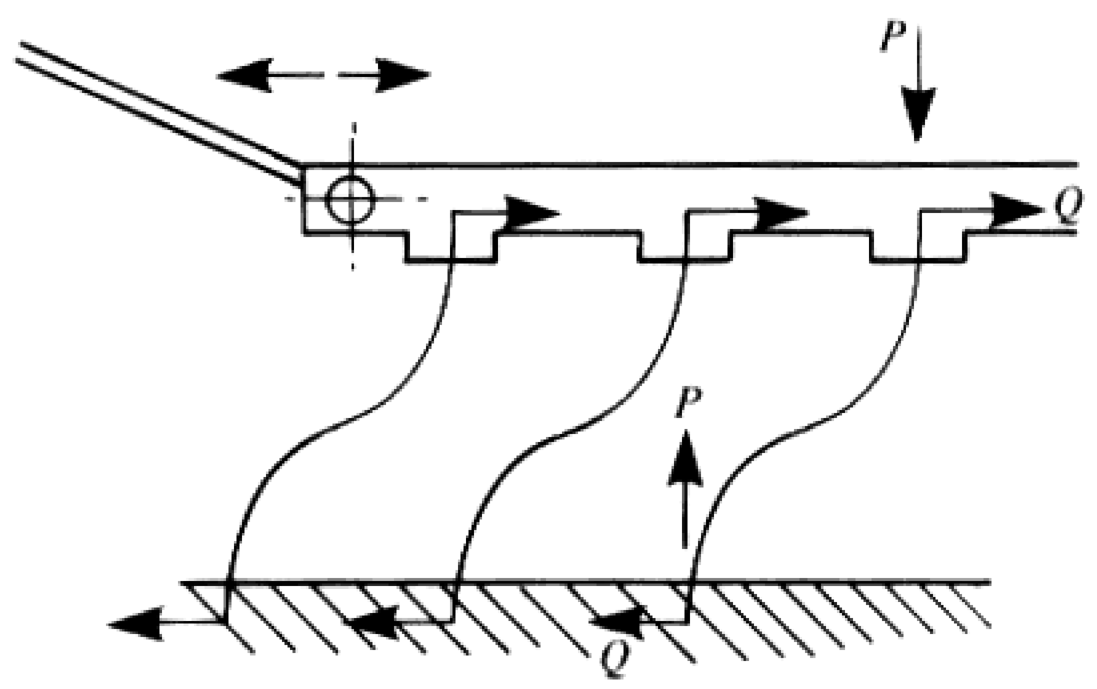

Fig. 29 Suspended vibrating screen

and the deflection by

$$\delta = \frac{2Q}{P\,[l - \{(\tan wl)/w\}]},$$

where $w = \sqrt{(12P/t^3bE)}$.

If, however, the screen is suspended from the lower end of the springs the force P becomes tensile and the stress is given by

$$f = +P/bt \pm \left(\frac{6Q \tan wl}{wbt^2}\right),$$

the sign of the result indicating whether the resultant stress is compressive (minus) or tensile (plus).

Laminated springs

It has been demonstrated that one of the most efficient of the cantilevers is the triangular, uniform-section spring. Two of these may be joined together base-to-base in the shape of a diamond, forming an efficient beam spring which may in theory be loaded as in Fig. 27. With the carriage spring (Fig. 27(b)), which is the ideal spring for vehicle suspension since its low volume represents a minimum increase in the unsprung weight, it is usually impossible to incorporate the width into the available space. This difficulty however may be overcome by a simple but ingenious procedure.

Suppose that calculations lead to the diamond-shaped spring shown in Fig. 30, where l is the overall

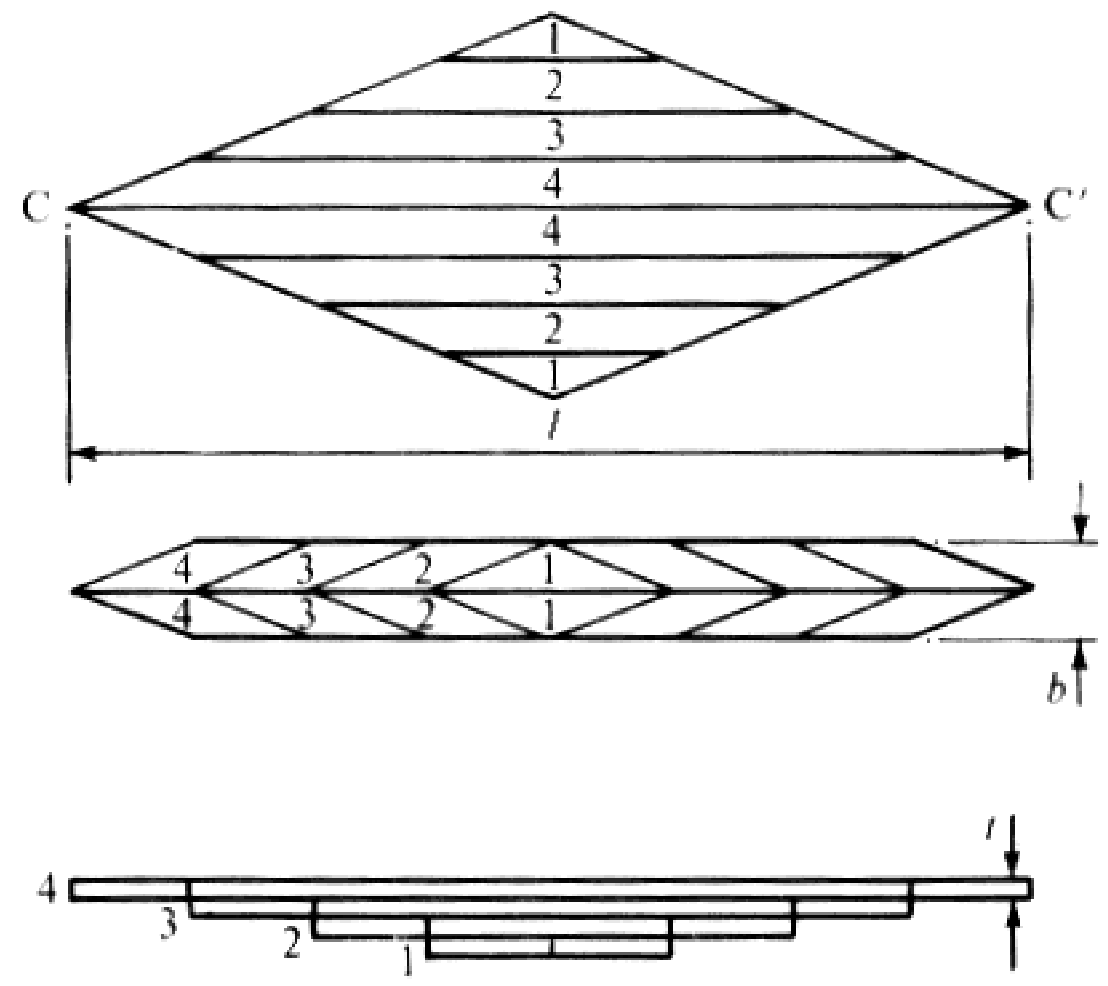

Fig. 30 Theoretical design of trapezoidal spring of uniform stress

length and b the overall width, then proceed as follows.

1. Divide the spring down the centre line CC′.
2. Divide each half into a suitable number of strips of equal width such that twice the strip width will comply with the space limitations of the design, and starting from the two outside portions number the strips 1,1; 2,2, etc.
3. Place 1 and 1 side by side to form a diamond; place 2 and 2 side by side centrally above 1 and 1, and continue until all the strips are used up.

This results in a laminated spring having the same physical characteristics as the original leaf. In practice, each section is a single leaf cut from a strip equal in width to two of the strips in the theoretical design.

This design has the same disadvantages as the triangular cantilever, since in practice it is impossible to apply a load at either of the pointed ends and, furthermore, the points at the ends of the intermediate leaves would damage the plate above them and give rise to excessive local stresses. Two possible ways of avoiding these disadvantages are (1) to leave the width of the strip constant throughout its length and taper the thickness according to a parabolic form, and (2) to forfeit some of the uniformity of stress by cropping the ends of each leaf to give a trapezoidal section. In practice method (1) is very expensive and most laminated leaf springs are now designed using a trapezoidal form in which the shorter side is equal to the finished width of the spring as shown in Fig. 31. Each remaining strip is one half of the width of the main leaf and the whole assembly is built up as before. Once again, in normal production, the leaves are cropped from strip of uniform width.

Some allowance must be made for the fact that the pack must be held together in the centre and this leads to a zone which is practically dead and contributes very little to the deflection under load. This portion must be added to the trapezoidal shape in the original design and is denoted by the hatched areas in Fig. 31.

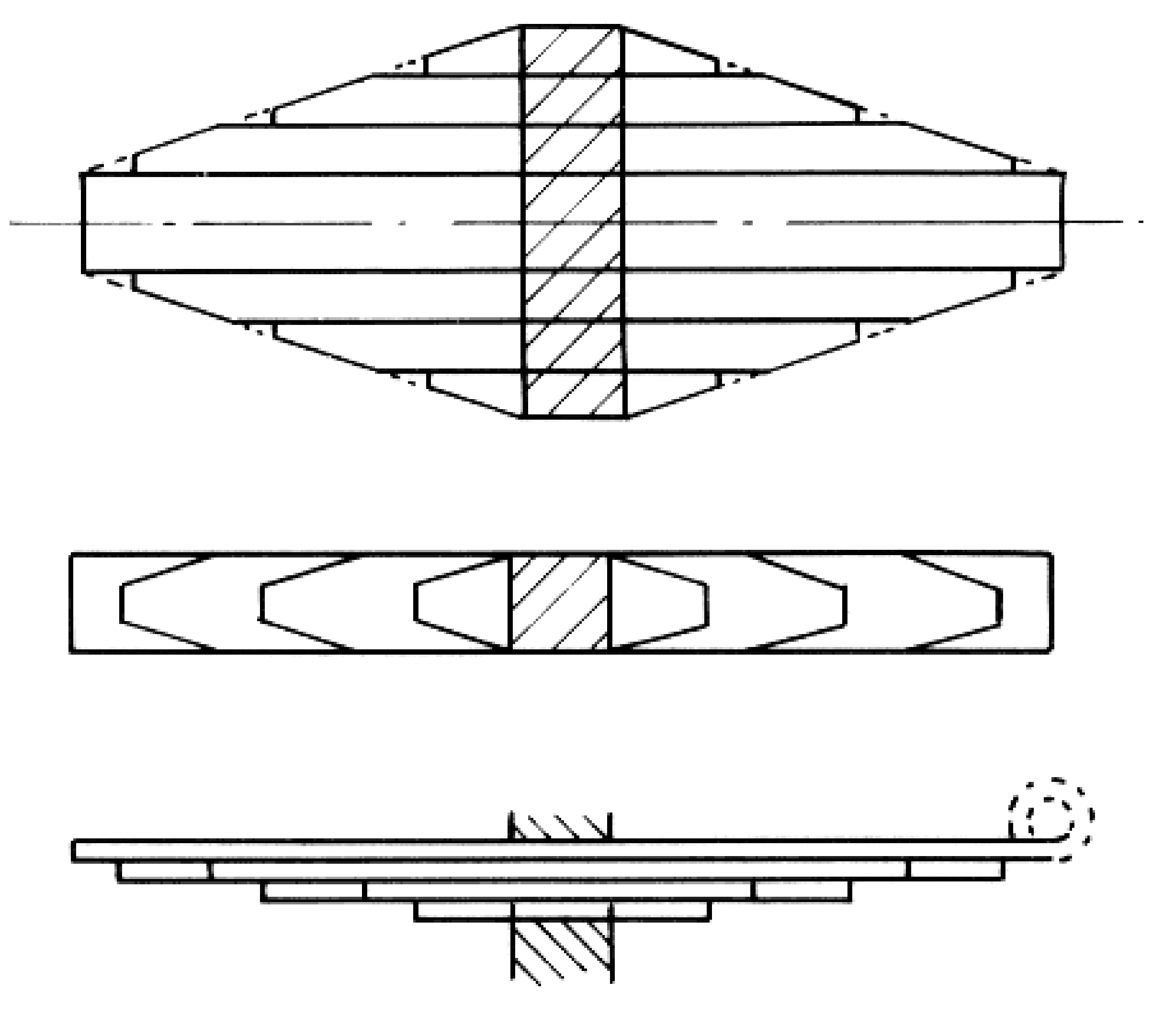

Fig. 31 Practical modification of theoretical trapezoidal spring

Where the spring is subject to sudden shock loading, the line of contact between the second longest leaf and the main leaf is an area of high local stress, and it is usual practice to start with a trapezoidal design in which the shorter side is twice or even three times the finished spring width. This main plate is then divided into two or three strips and the assembled spring has two or three top leaves of equal length.

Worked example

It is required to design a laminated leaf spring of the trapezoidal type in which the distance between load-bearing points is 530 mm with a central location of 30 mm. The maximum load is 7000 N and the available space to accommodate the width is 50 mm. The full deflection under load is 33 mm at which point the stress should not exceed 900 N/mm².

Calculation. By subtracting the location from the total length and dividing the spring into two cantilevers, it is found that $l = 250$ mm.

The maximum strip thickness of a rectangular leaf which will deflect 33 mm at a stress of 900 N/mm² is given by:

$$t = \frac{2l^2 f}{3E\delta} = \frac{2 \times 250^2 \times 900}{3 \times 207\,000 \times 33} = 5.5 \text{ mm}.$$

This is the *maximum* but, to ease calculations, use 5.0 mm and adjust later.

The width of 5 mm strip to carry 3500 N at 900 N/mm² is given by:

$$b = \frac{6Pl}{t^2 f} = \frac{6 \times 3500 \times 250}{5^2 \times 900} = 233 \text{ mm}.$$

Since a trapezoidal spring is proposed, this figure may be rounded down to 200 mm in view of the more advantageous stress distribution.

The width of the narrow end must accommodate two strips, each of which must be less than 50 mm. Starting with a width of 80 mm, then the shape ratio, m, is 40 : 200, i.e. 0.2, whence, $M = 1.2$ and the limits of the spring can be drawn (see Fig. 32). It can be seen that the strip width is 40 mm and a total of 5 leaves can be cut from the blank.

Check. The design is now 2 top leaves (full length) and 3 supplementary leaves 40 mm × 5 mm. Thus:

$$\delta = \frac{1.2 \times 3500 \times 250^3 \times 4}{207\,000 \times 200 \times 5^3} = 42 \text{ mm},$$

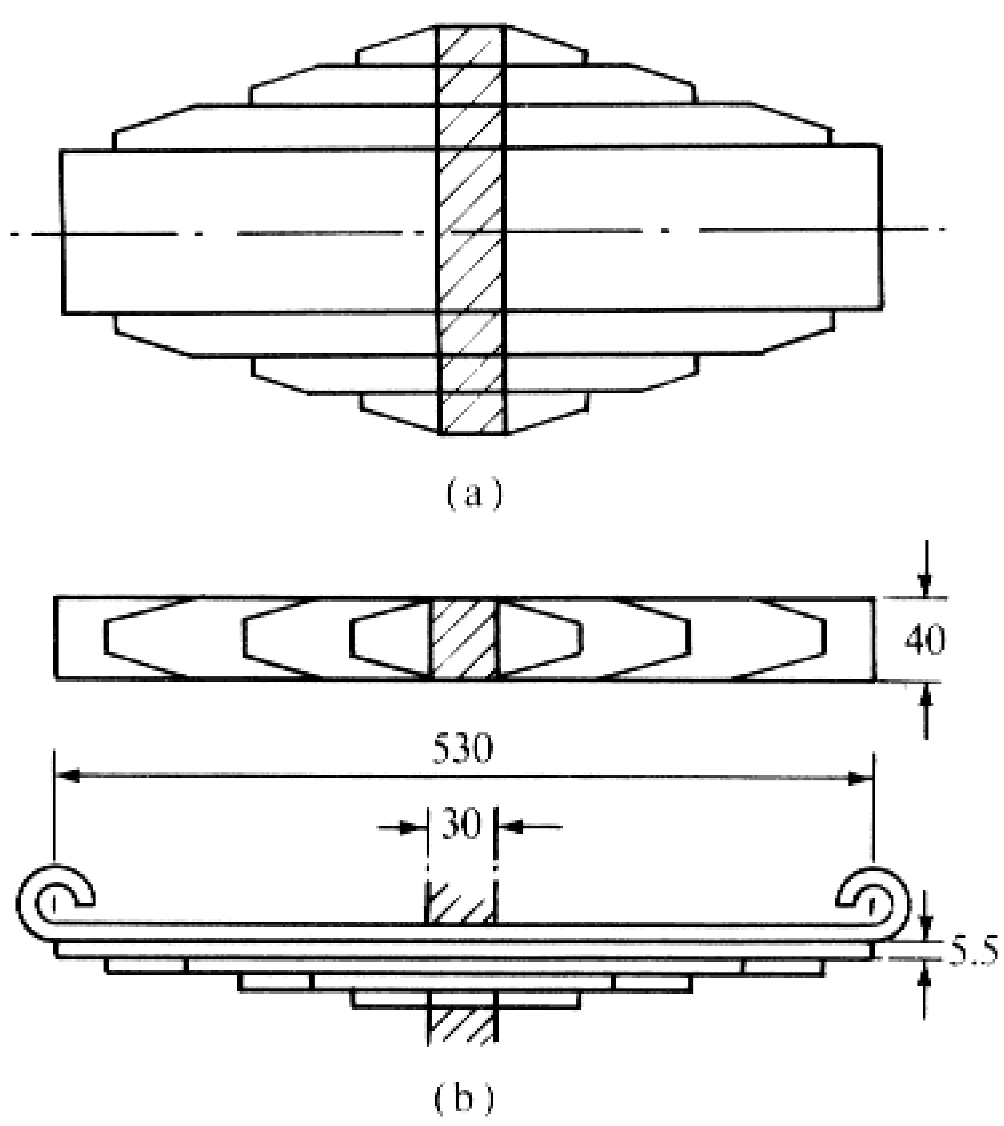

Fig. 32 Solution to worked example on trapezoidal spring (all dimensions in mm)

indicating that the leaf is too thin. The thickness of the leaf may be adjusted to a final value t_1 as follows:

$$(t_1/t)^3 = 42/33 = 1.273$$

thus

$$t_1 = 1.273^{\frac{1}{3}} \times 5 = 5.4 \text{ mm}.$$

Stress at full load,

$$f_{max} = \frac{6Pl}{bt_1^2} = \frac{6 \times 3500 \times 250}{200 \times 5.4^2} = 900 \text{ N/mm}^2.$$

The final spring assembly is shown in Fig. 32.

Springs stressed in bending

Spiral and power springs

Springs coiled from flat strip in the form of a two-dimensional spiral are generally known as *spiral springs*. If during operation the coils remain entirely free of each other, the term *hair spring* is often applied. If however the spring is tightly wound so that all coils are in contact round the arbor and delivers its energy by unwinding until all coils are closely packed inside a retaining box, it is commonly known as a *clock spring* or *motor spring*.

Hair springs

Hair springs may be divided roughly into the following classes:

(1) those having a large number of coils with a high shape ratio; these can be subdivided into
 (i) clamped springs, in which the outer end is rigidly clamped, and
 (ii) pinned-outer-end springs, which are free to assume a position of minimum restriction under load;
(2) those having a few coils ($\leqslant 2\frac{1}{2}$) with a low shape ratio, $b/t (< 10)$.

It is not usual to pin the outer end of a class (2) spring.

Many turns, clamped outer end. Fig. 33 shows a hair spring having a large number of coils with clamped outer end.

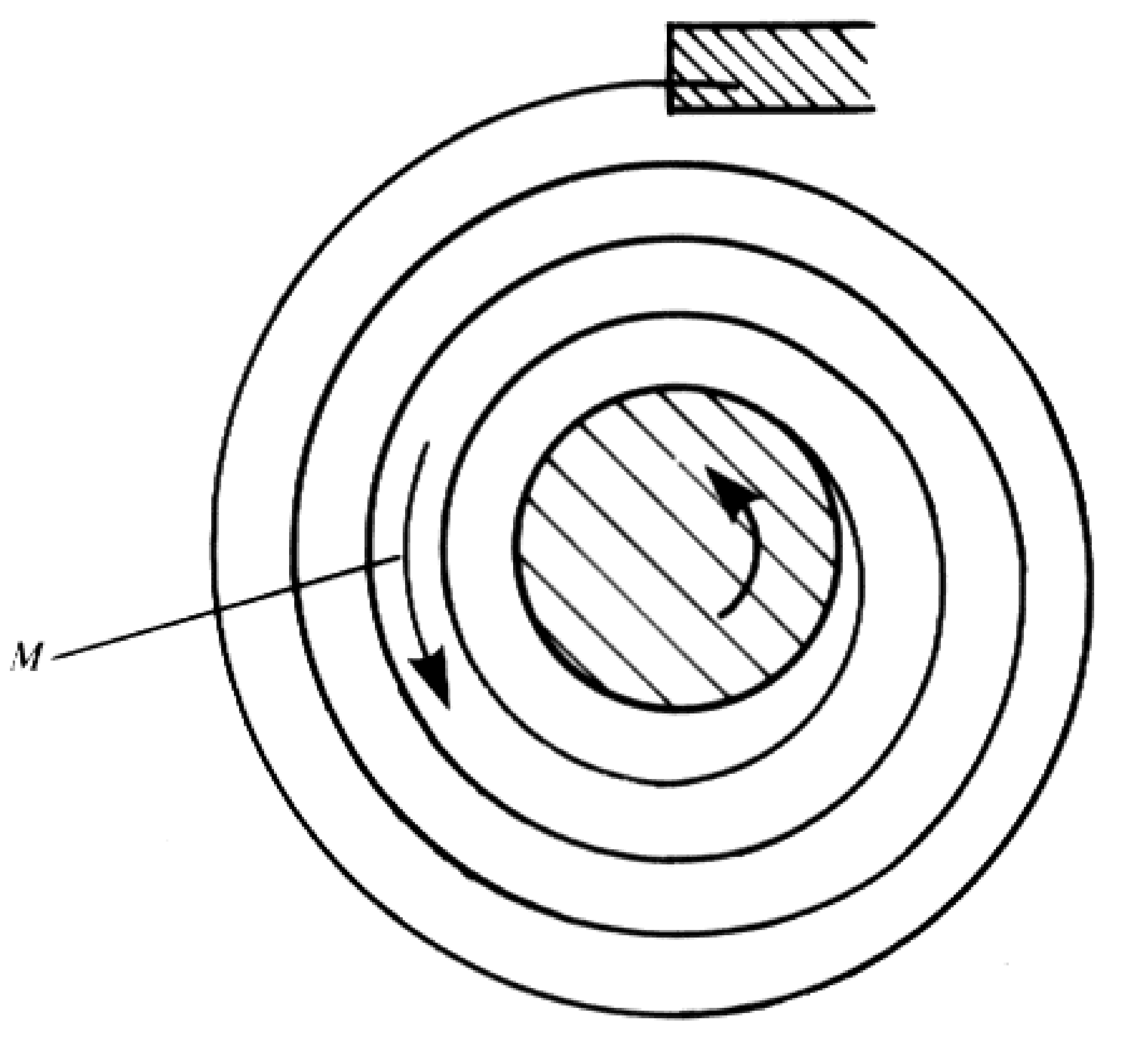

Fig. 33 Hair-spring with clamped outer end and many coils

If b is strip width, t is strip thickness, l is active length of the strip in the spring, E is Young's modulus, M is operating moment, n is number of turns (each of 360°) produced by torque M, and f is the bending stress (uniform throughout strip), then:

$$n = 6Ml/\pi Et^3b, \qquad (15)$$

and

$$f = 6M/t^2b. \qquad (16)$$

Combining eqns (15) and (16):

$$f/n = \pi Et/l,$$

whence

$$f = \pi Etn/l. \qquad (17)$$

If at any point it is necessary to crank the material, particularly close to either end, a correction factor based on the shape ratio D/t (where D is twice the mean radius of curvature, r, of the crank) should be applied. Fig. 34 shows the stress-concentration in graph form.

Many turns, pinned outer end. If the outer end of a strip is fitted over a pin, the only restraining moment at this point is frictional and the corresponding expression for deflection becomes:

$$n = 15Ml/2\pi Et^3b. \qquad (18)$$

Thus for the same torque a pinned outer end spring moves 25 per cent further than one with a clamped end. The stress in such a spring is given by

$$f = 12M/t^2b, \qquad (19)$$

which is twice that for a clamped spring.

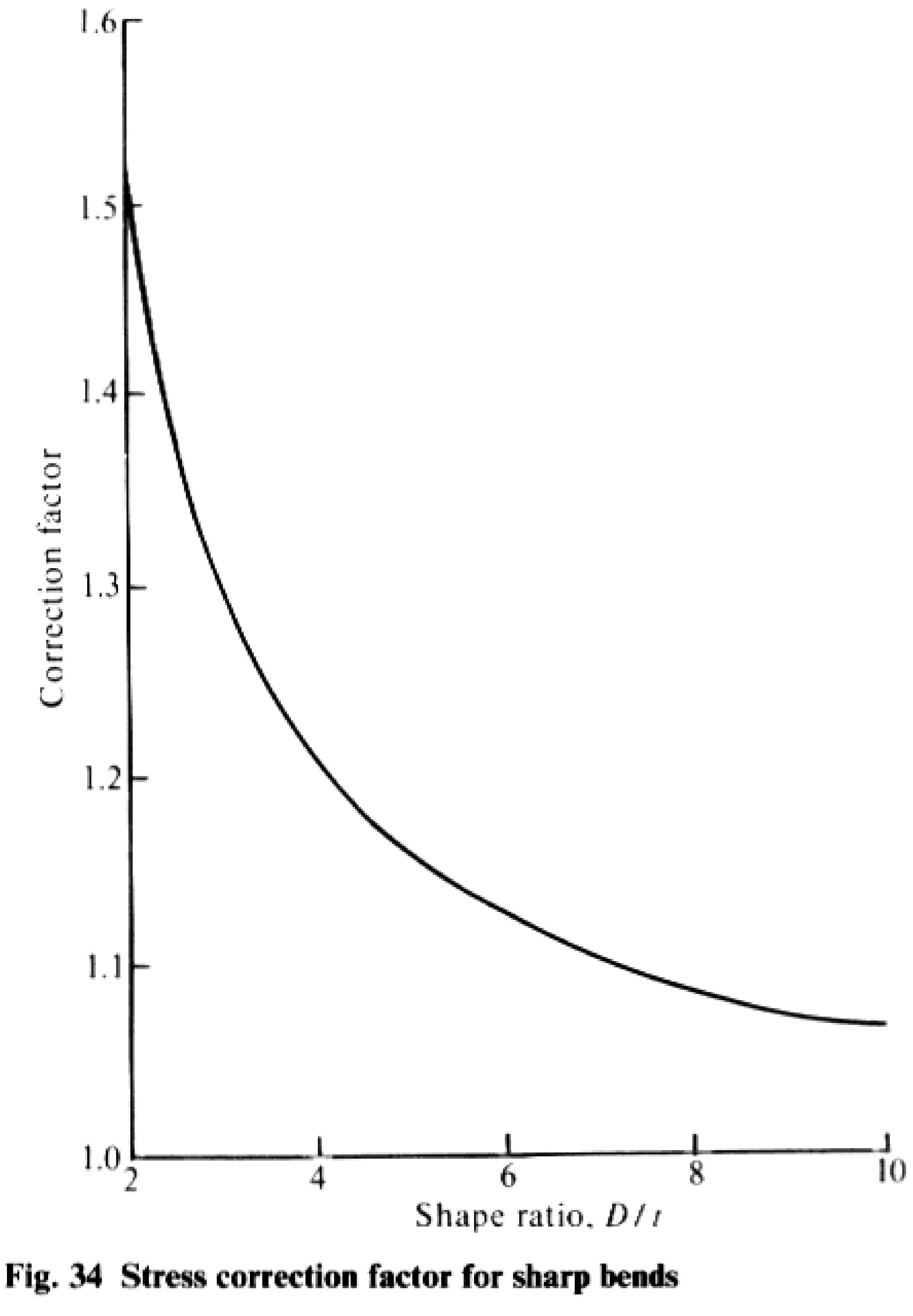

Fig. 34 Stress correction factor for sharp bends

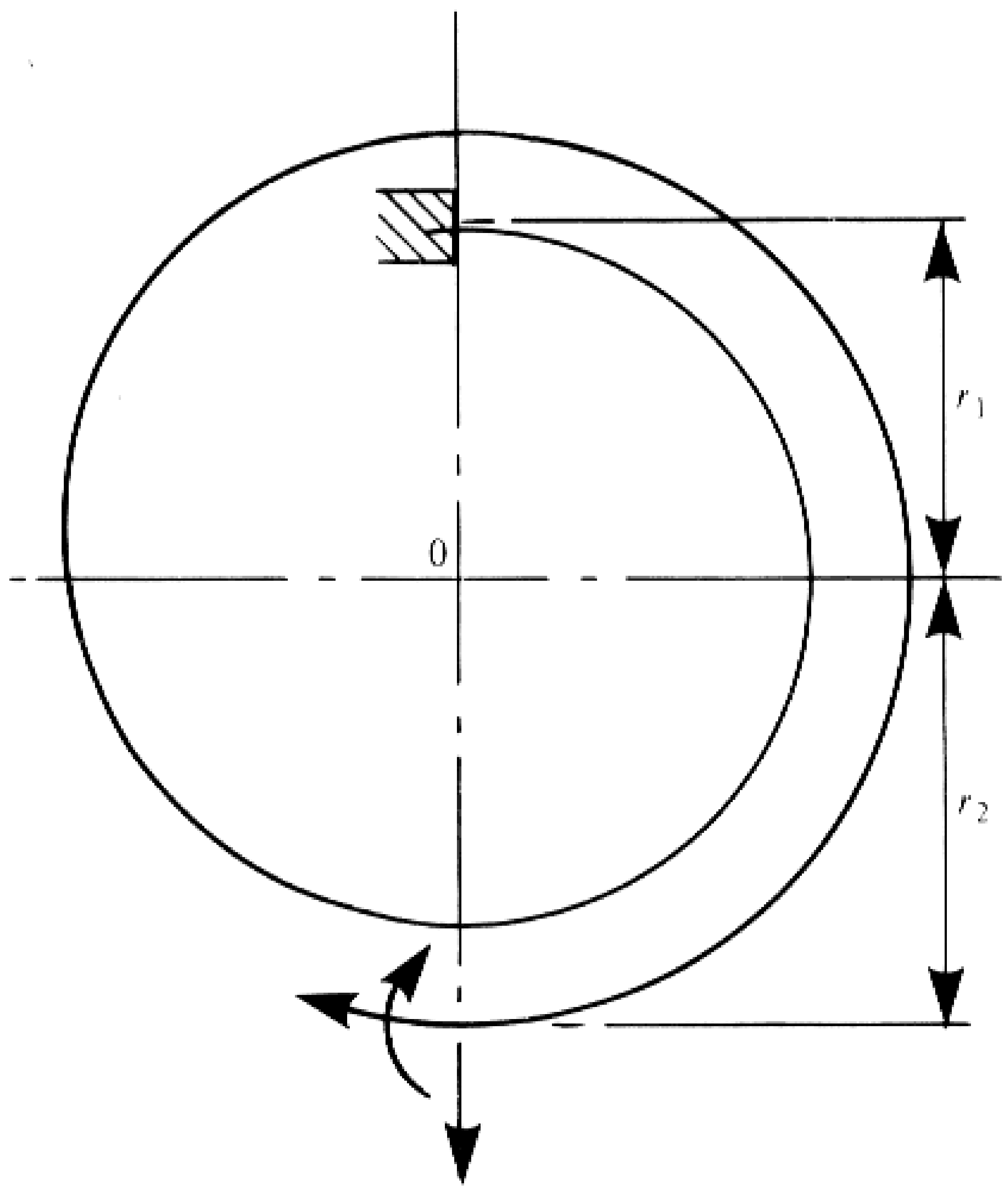

Fig. 35 Hair-spring with clamped outer end and few coils: O is centre of spring

Few turns, clamped outer end. In a hair spring having few coils and with the outer end clamped and restrained to move in an arc about the centre of the spring (see Fig. 35), it is necessary to modify the maximum stress by a factor α, which depends upon the actual shape of the spring and the number of degrees, θ, traced out by the radius vector in moving from one end of the strip to the other. The shape factor λ is defined as:

$$\lambda = l - (r_1/r_2),$$

where r_1 is the radius at the inner end and r_2 the radius at the outer end.

Values of α are shown plotted against θ for various values of λ in Fig. 36.

In addition the stiffness (movement per unit of applied torque) has to be modified by a factor β which is shown plotted against the same values of λ in Fig. 37.

Springs of this type are usually found as return or balance springs in precision instruments and for this purpose it is essential that their physical properties

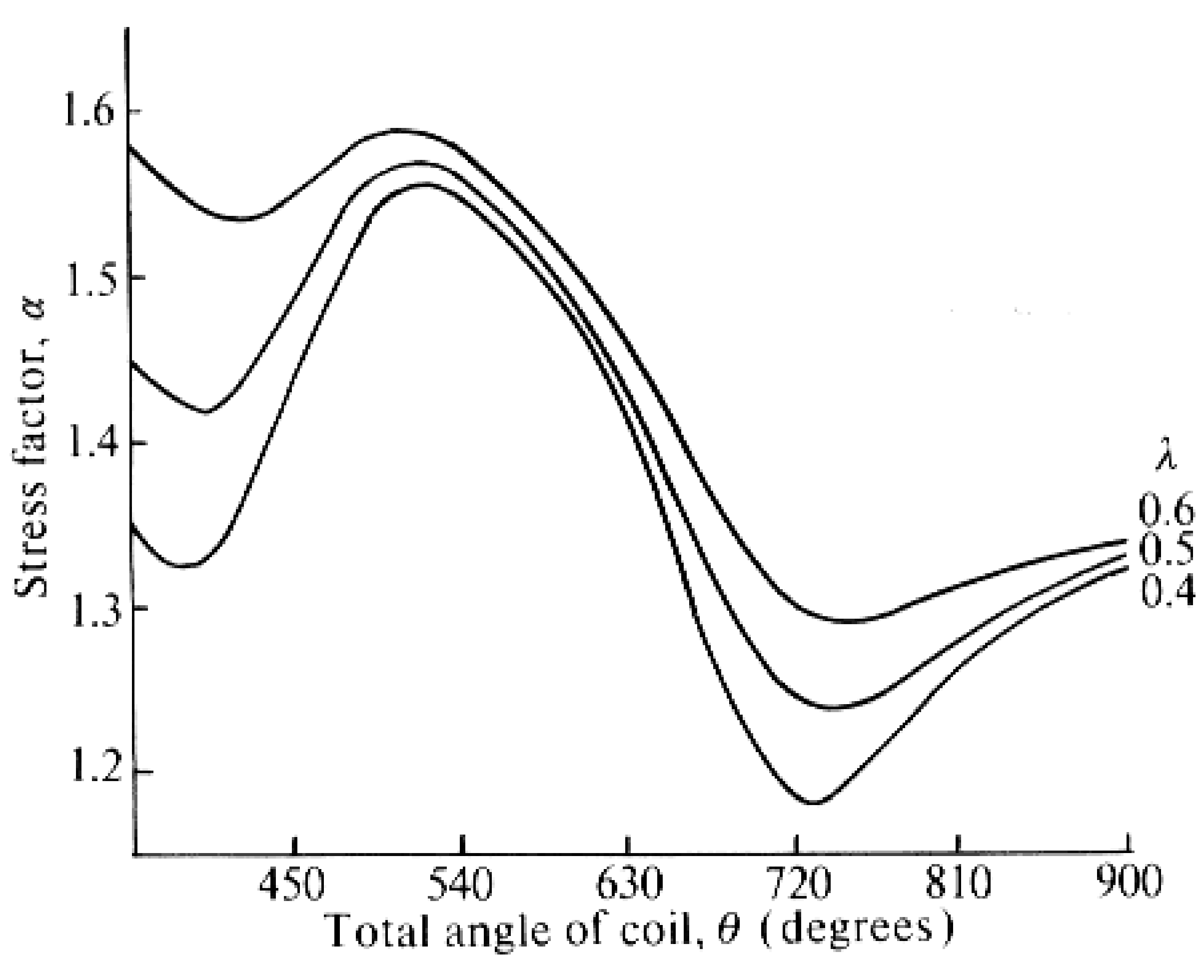

Fig. 36 Stress concentration factor, α, for spring with few coils; $\lambda = 1 - (r_1/r_2)$

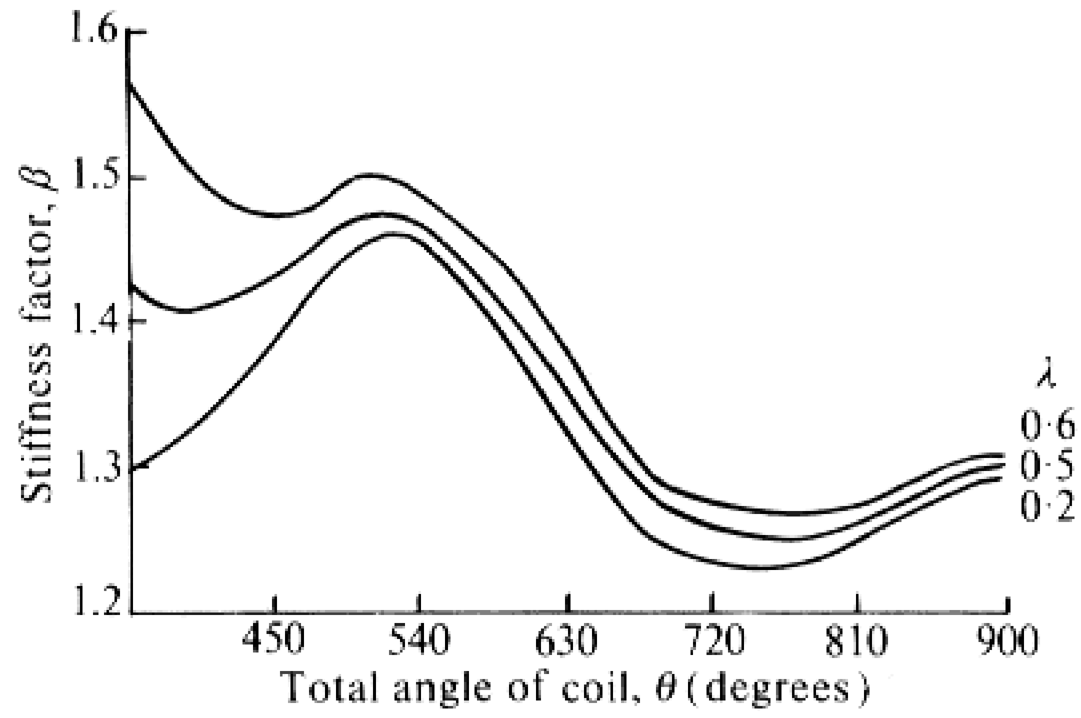

Fig. 37 Stiffness factor, β, for spring with few coils; $\lambda = 1 - (r_1/r_2)$

are suitable for the particular application. Where it is essential that the modulus of elasticity does not change appreciably over the range of naturally occurring temperatures (−45°C to 65°C) precipitation-hardening Ni-Span C is used, while Inconel is commonly used if the instrument is liable to be affected by magnetism. Other materials suitable for the manufacture of hair springs are high-carbon steel, phosphor-bronze, beryllium copper, nickel silver, austenitic stainless, and many of the silver alloys.

Permissible stress. Table 2 gives the approximate maximum stress to which balance springs of this type should be subjected, together with the stress range permitted if the spring is subject to oscillation.

Table 2. Permissible maximum stress and stress range for oscillating springs having few turns and clamped outer end

Material	Maximum stress	Stress range
	(N/mm^2)	(N/mm^2)
Carbon steel	550	175
Phosphor bronze	420	175
Beryllium copper	420	140
18/8 stainless steel	520	175
Ni-Span C	550	140
Silver	100	35

Clock or motor springs

The clock or motor spring has two main idealized shapes for which various mathematical theories have been expounded: (1) the *Archimedian spiral* in which the space between the coils in their free state is constant, and (2) a logarithmic form in which the radius of curvature is directly proportional to the distance from the centre of the spring. In practice neither of these forms is easily obtained, although a spring coiled from hard-rolled strip and wound tightly on its arbor will spring open to form a reasonably close approximation to the latter. A low-temperature heat treatment of approximately 260°C will stabilize the shape. In order to provide anchorage within the case it is usual to provide a hole at the outer end to engage with an undercut pin, and a right-angle crank at the inner end to engage in a slot cut in the arbor (see Fig. 38). This necessitates annealing both ends of the strip and great care must be exercised to ensure close control of the length annealed and a gradual transition from the soft to the hard zone. Typical examples of this type of spring are the gramophone motor spring and the telephone-dial return spring, both of which are required to provide a constant torque throughout their ranges of operation.

Unfortunately this type of spring delivers its energy in an erratic manner governed not only by the exact conformation of the spring in its free state (which is difficult to control in practice) but also by the inter-coil friction which is present when the spring is wound. Fig. 39 shows a typical torque–revolutions curve for such a spring. It will be noted that the curve is relatively steep over the first 25 per cent of its wind-up, during which about 55 per cent of the maximum torque is achieved. Thereafter the curve flattens to give a relatively straight line with approximately one third of the rate of the first portion. During wind-up the pressure between adjacent coils produces frictional forces that prevent uniform sliding motion, and such motion as occurs takes place in sudden jerks.

Since motor springs invariably operate only in the unwinding mode, hysteresis must be added to the other variables. Various devices have to be adopted in practice in order to make the rate of energy delivery acceptable. With the gramophone spring it is consistent table speed that is necessary, and this is achieved by fitting a governor and friction brake.

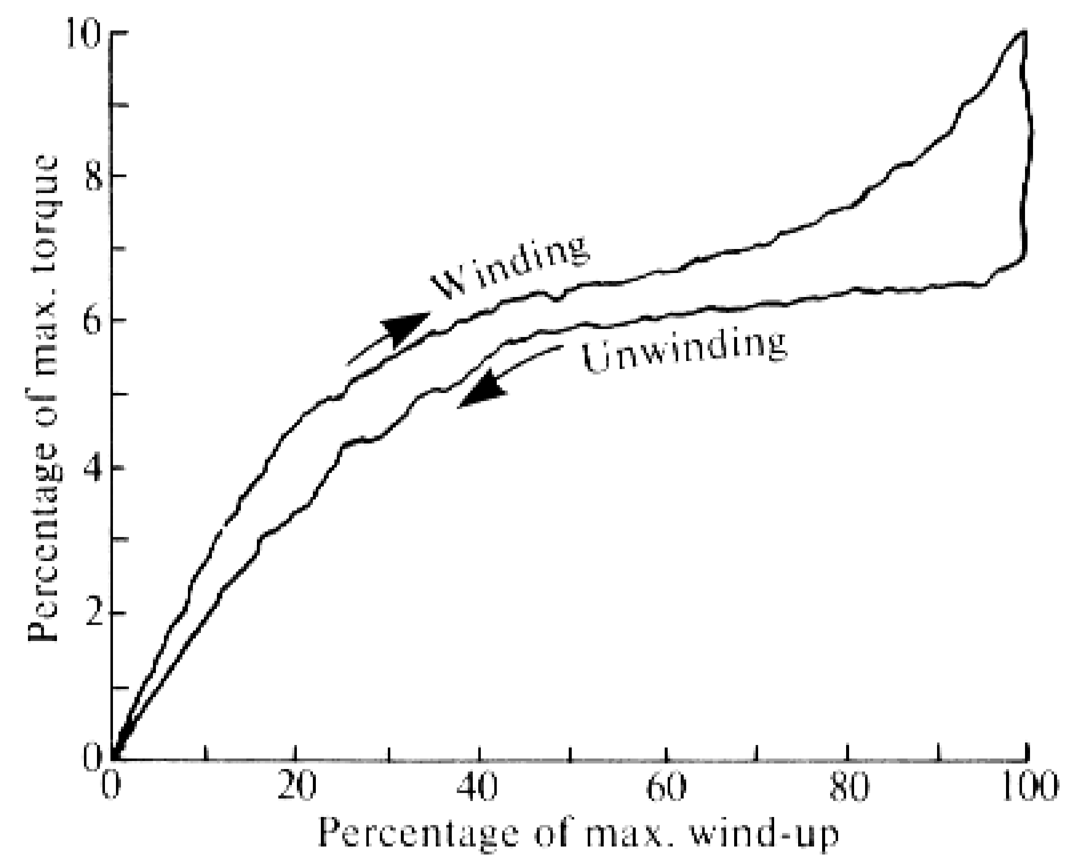

Fig. 39 Typical torque–revolution curve for cased spiral spring lubricated with molybdenum disulphide

This of course means that the energy dissipated through friction in the form of heat is wasted so that the motor is rather inefficient, but it does provide a means of speed adjustment. In the case of the telephone-dial spring the object is achieved by using only a small portion of the top end of the curve. The spring is capable of accepting six or more complete revolutions but the dial is restricted to a total movement of considerably less than a single revolution. The design of the case enables the mechanism to use the most advantageous portion of the curve.

Design

Referring to Fig. 38, where D_c is the inside diameter of the case, D_u is the inside diameter of the unwound spring in the case, D_w is the outside diameter of the

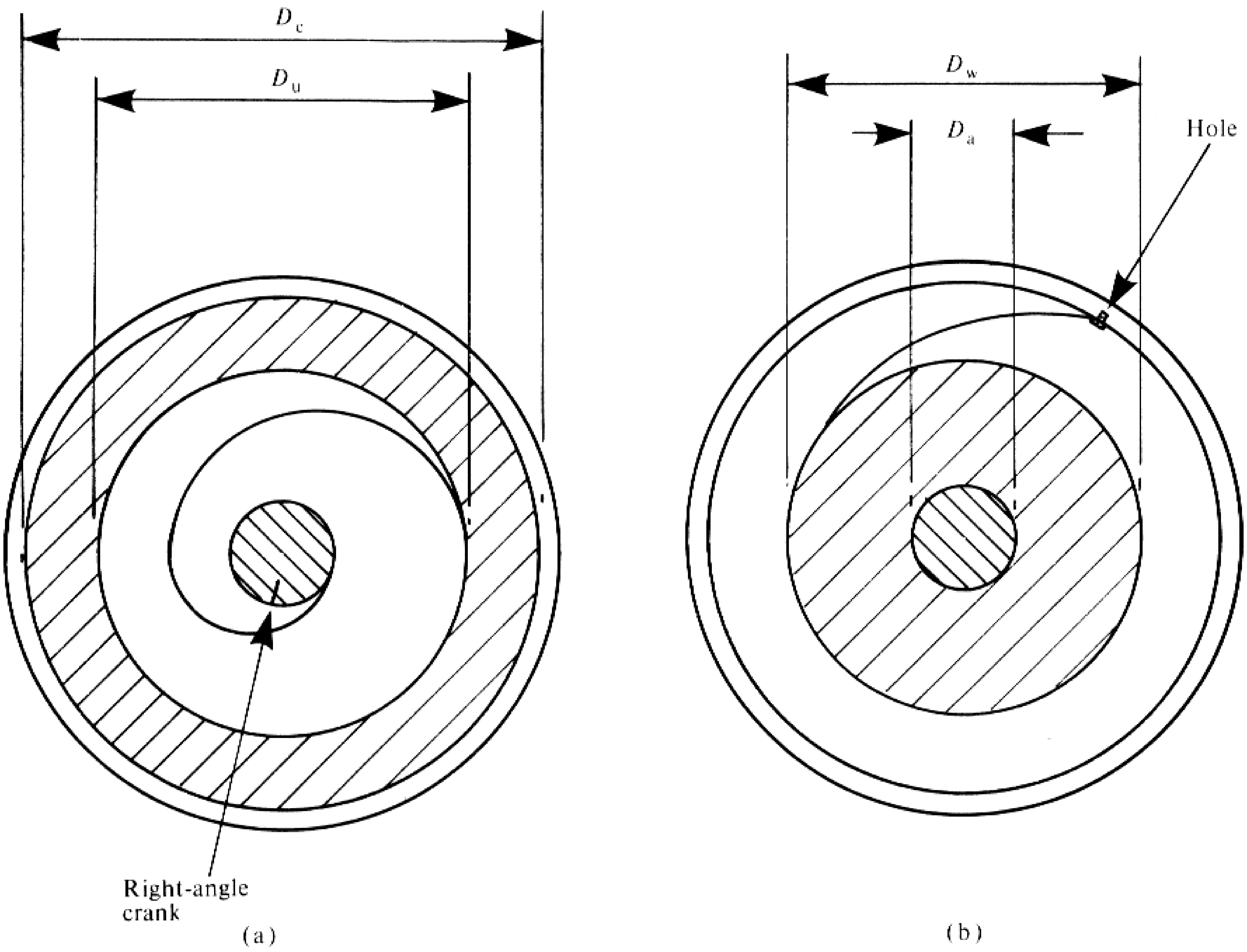

Fig. 38 End fixings for motor spring: (a) unwound in case; (b) fully wound

wound spring on the arbor, D_a is the diameter of arbor, n_w is the number of coils in the wound spring and n_u is the number of coils in the unwound spring, and assuming that in both the wound and the unwound conditions there is no space between the coils, then the number of turns that such a spring will deliver from the various cross-sectional areas may be estimated.

Thus, ignoring the length of material which connects the wound spring to the case, cross-sectional area of material = lt, and cross-sectional area of wound spring = $\frac{1}{4}\pi(D_w^2 - D_a^2)$, whence

$$D_w = \{D_a^2 + (4lt/\pi)\}^{\frac{1}{2}} \qquad (20)$$

and

$$2n_w t = D_w - D_a, \qquad (21)$$

giving

$$2n_w t = \{(4lt/\pi) + D_a^2\}^{\frac{1}{2}} - D_a \qquad (22)$$

and

$$n_w = [\{(4lt/\pi) + D_a^2\}^{\frac{1}{2}} - D_a] \div 2t. \qquad (23)$$

Considering the unwound spring in the same manner:

$$n_u = (D_c - D_u)/2t \qquad (24)$$

and

$$lt = \tfrac{1}{4}\pi(D_c^2 - D_u^2), \qquad (25)$$

whence

$$D_u = \{D_c^2 - (4lt/\pi)\}^{\frac{1}{2}}, \qquad (26)$$

and

$$n_u = [D_c - \{D_c^2 - (4lt/\pi)\}^{\frac{1}{2}}]/2t. \qquad (27)$$

The number of coils delivered is therefore the difference between n_u and n_w.

$$n_w - n_u = [\{D_a^2 + (4lt/\pi)\}^{\frac{1}{2}} + \{D_c^2 - (4lt/\pi)\}^{\frac{1}{2}} - (D_c + D_a)]/2t. \qquad (28)$$

It will be appreciated that this is an idealized maximum which assumes that the strip is perfectly flat and that there is perfect contact between adjacent coils. This of course is never achieved.

As a result of analysis of many hundreds of practical designs it has long been established that if a spring is to deliver the maximum number of turns, then the area of the wound-up spring should be approximately one half of the available area between the arbor and the case, which leads to the condition that

$$l = (D_c^2 - D_a^2)/2.55t. \qquad (29)$$

Substituting this in eqn (28) and simplifying, then

$$n_w - n_u = (D_c^2 - D_a^2)/2tu = 4l/\pi U, \qquad (30)$$

where

$$U = \frac{D_w^2 - D_a^2}{(2D_c^2 + 2D_a^2)^{\frac{1}{2}} - (D_c + D_a)}.$$

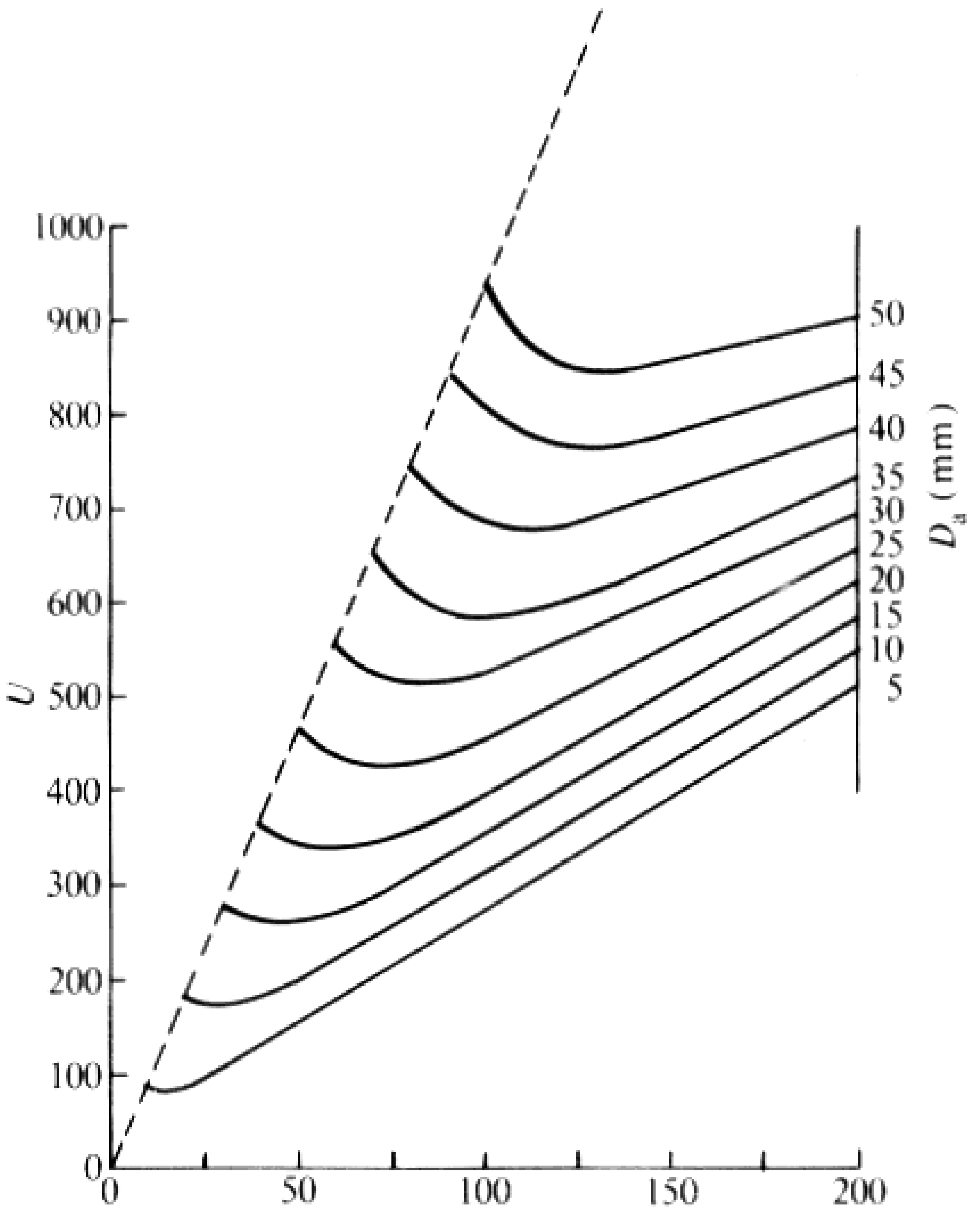

Fig. 40 Design factor for number of turns delivered

For ease of design the expression for U is given graphically in Fig. 40 for values of $D_c \leqslant 20$ mm and values of $D_a \leqslant 50$ mm. It should be stressed here that Fig. 40 only gives the correct value for U if all dimensions are measured in millimetres.

Stress

Assuming a uniform distribution of stress along the length of the spring the formula connecting stress and torque is given by

$$f = 6Mt^2b,$$

which is the same as eqn (16) in the section on hair springs.

Fig. 41 shows recommended maximum stresses for various thicknesses of high-carbon steel strip.

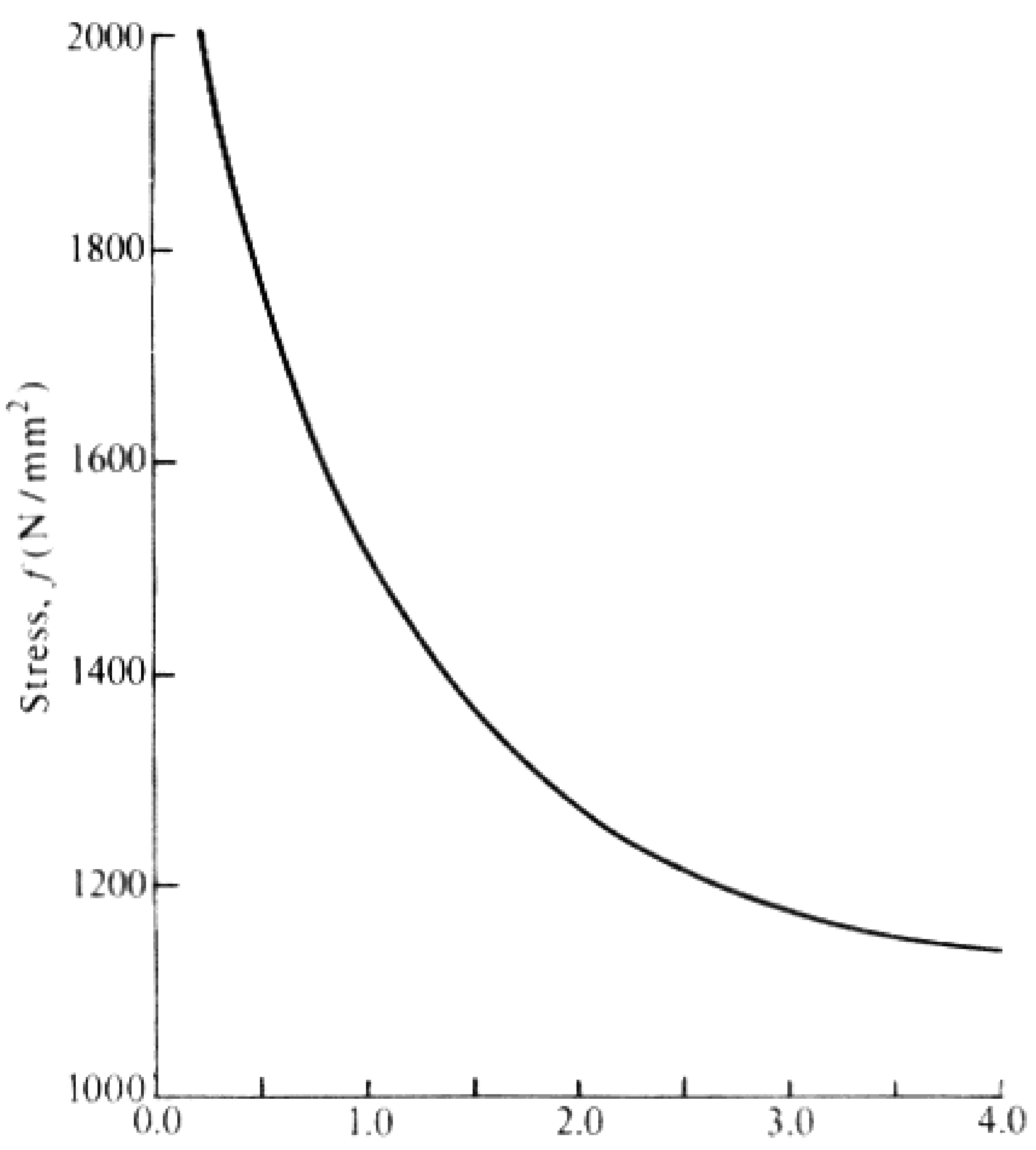

Fig. 41 Recommended maximum stresses for various thicknesses of high-carbon spring steel

Using the formulae thus far established the relevant design relationships may now be summarized as follows:

(i) $f = 6M/t^2b$

(ii) $f = \pi Et(n_w - n_u)/l$

(iii) $n_w - n_u = fl/\pi Et$

(iv) $t^3 = 1.5MU/Eb$

(v) $l = (D_c^2 - D_a^2)/2.55t$.

Worked example

It is desired to design a motor spring to fit into a case 100 mm inside diameter, that will accept a strip width of 30 mm and has an arbor of 18 mm diameter. The maximum torque required is 4800 N mm and the spring should deliver approximately 15 turns.

From Fig. 40, $U = 380$, whence

$$t^3 = \frac{1.5 \times 4800 \times 380}{207\,000 \times 30} = 0.441,$$

and

$$t = 0.76 \text{ mm}.$$

From (v), $$l = \frac{10\,000 - 324}{2.55 \times 0.76} = 4992 \text{ mm}.$$

From (i), $$f = \frac{6 \times 4800}{30 \times 0.76^2} = 1662 \text{ N/mm}^2,$$

which is satisfactory according to Fig. 40.

From (iii), number of turns delivered,

$$n_w - n_u = \frac{1662 \times 4992}{\pi \times 207\,000 \times 0.76} = 16.8 \text{ turns}.$$

Because of hysteresis and mechanical friction it is unlikely that the spring will deliver its last turn or two and this figure is close enough.

Practical guidelines. When designing and manufacturing motor springs the following practical guidelines have emerged from past experience:

(1) the arbor diameter should be larger than 20 times the material thickness, or stresses may be excessive;

(2) the length of strip should not exceed 15 000 times its own thickness or the torque–revolution curve will be excessively distorted by inter-coil friction and 'bundling' of the coils (a condition where the coils do not peel off uniformly but come away in bundles of two or three at a time);

(3) the cross-section of the edges of the strip should be as nearly semi-circular as possible in order to obtain the best fatigue performance, and

(4) the innermost coil should be continued to produce at least one complete dead coil of slightly less diameter than the arbor.

Constant-force spring

In order to overcome the disadvantages associated with the orthodox type of motor spring, an ingenious method of utilizing the characteristics of very thin strip was developed in the early 1950s in the USA. These so-called *zero-rated* springs are marketed under the names Tensator and Negator in the UK and USA, respectively. The design and manufacture of such springs is of a very specialist nature, and the accumulated experience of the originators is such that it is hardly worthwhile carrying out the full design process for a new design as it is almost certain that a combination of stock springs may be assembled to meet the specification.

Zero-rated springs appear in two standard forms: the constant-force spring, or Tensator extension spring, and the constant-torque spring-motor described in the next section. The constant-force spring consists of a tightly wound spiral of thin strip designed so that it may be pulled out straight and then return to its original coiled form. Such a spring, which is shown in Fig. 42, may be used as a counterbalance on machine tools, drop windows, etc.

If P is the load to be supported, b the width of material, t the thickness of material, D_i the inside diameter of coil in its free state, D_n the maximum outside diameter of coil in its free state, and E the modulus of elasticity of material, then

$$P = \frac{Et^3b}{6.6}\left\{\frac{1}{D_n^2} - \left(\frac{1}{D_n} - \frac{1}{D_i}\right)\right\}.$$

Design guidelines. For springs with a small number of coils ($\leqslant 10$):

(1) $t \nless 26.4P/ES^2b$, where S is a stress factor related to expected life (see Fig. 43);

(2) $D_i = (Ebt^3/6.6P)^{\frac{1}{2}}$;

(3) $b = 26.4P/ES^2t$;

(4) when the spring is assembled into the mechanism, it is fitted over a bush which is of

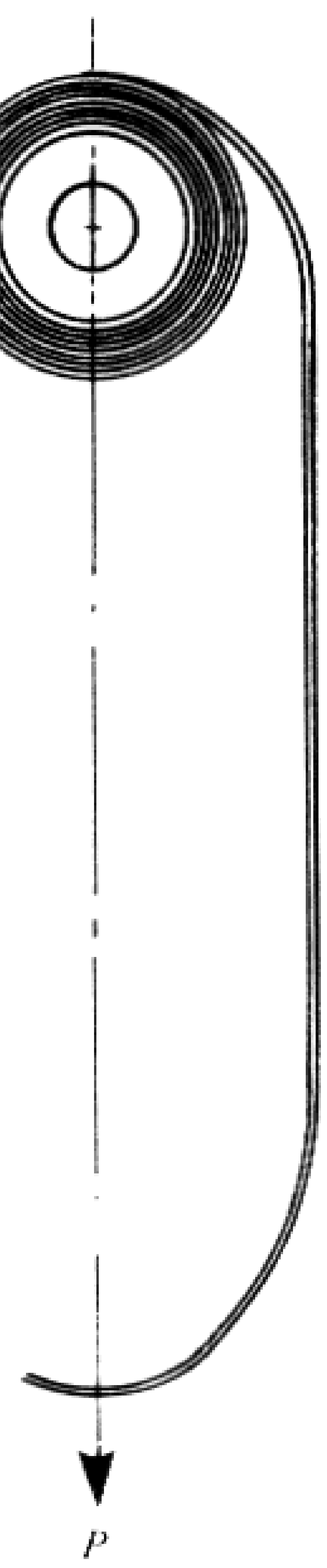

Fig. 42 Constant-force spring

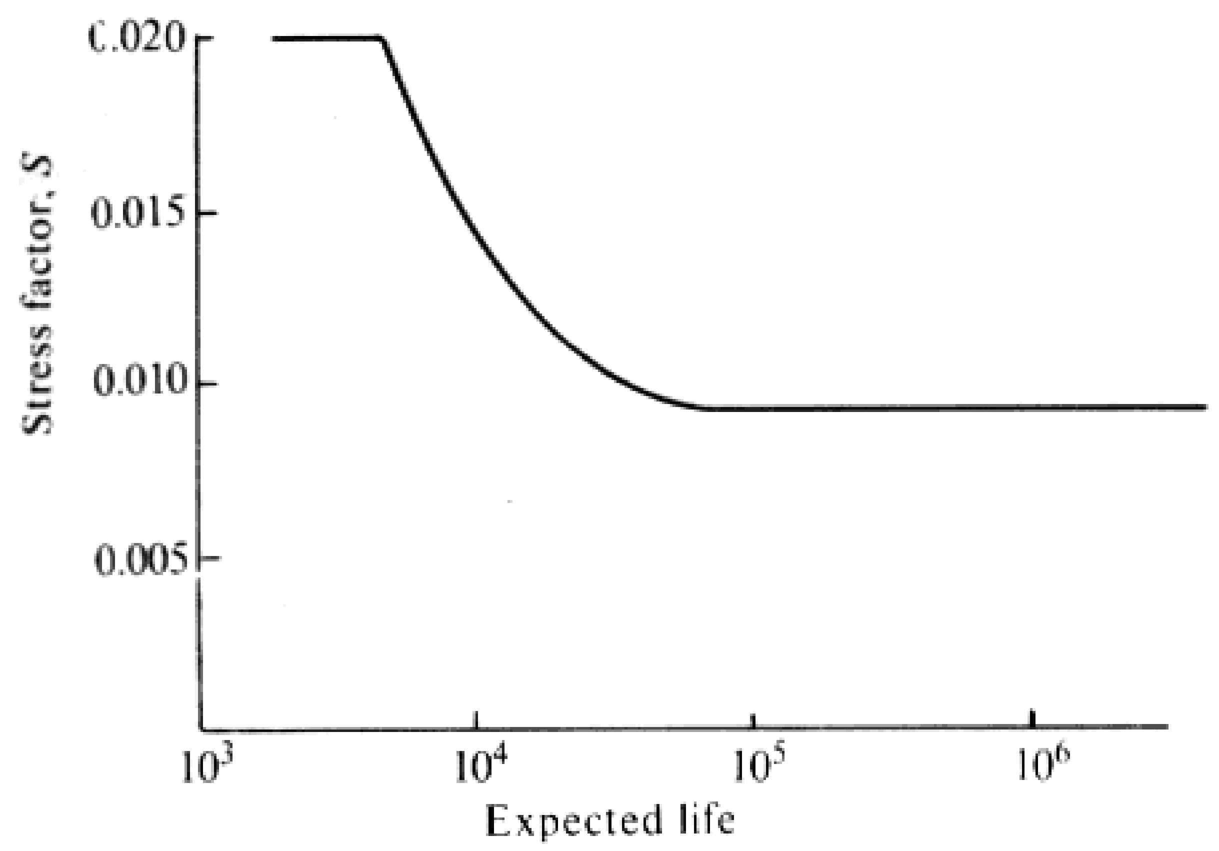

Fig. 43 Stress factor, *S*, for expected life of good-quality carbon spring steel

diameter D_b slightly greater than D_i, whence $D_b = 1.15\ D_i$, and

(5) length of strip in spring = deflection required plus $5D_b$, i.e. $l = \delta + 5D_b$.

Fig. 43 shows the recommended relationship between expected life and the stress factor, *S*, for good-quality carbon spring steel.

For springs with more than 10 coils:

(1) $t \nless 26.4P/ES^2b$;

(2) $D_n = (Et^3b/6.6P)^{\frac{1}{2}}$;

(3) $D_i = D_n/1.15$;

(4) $D_b = 1.15D_n$, and

(5) $l = \delta + 5D_b$.

Constant-torque motor springs

If a constant-force spring is placed upon a mandrel which allows it to turn freely and is then wound on to a larger mandrel, as in either Fig. 44 or Fig. 45, it will be found that the winding torque is almost constant over the middle 85 per cent of the wind-up. When released the spring will recoil itself on to the smaller mandrel, thereby imparting energy to the larger mandrel. The arrangement shown in Fig. 44 should be used where the major concern is length of life. That shown in Fig. 45 gives a higher torque for the same design of spring but at the expense of life.

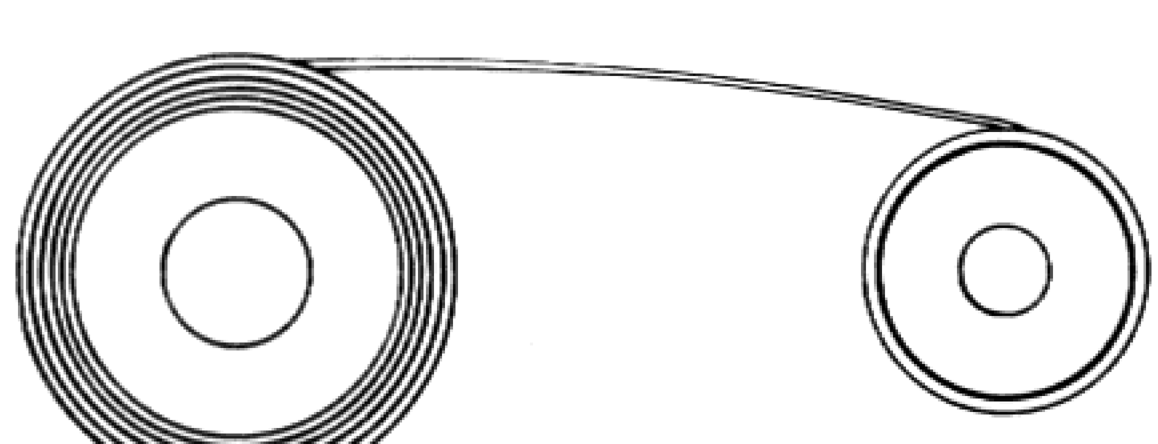

Fig. 44 Motor spring designed for long life

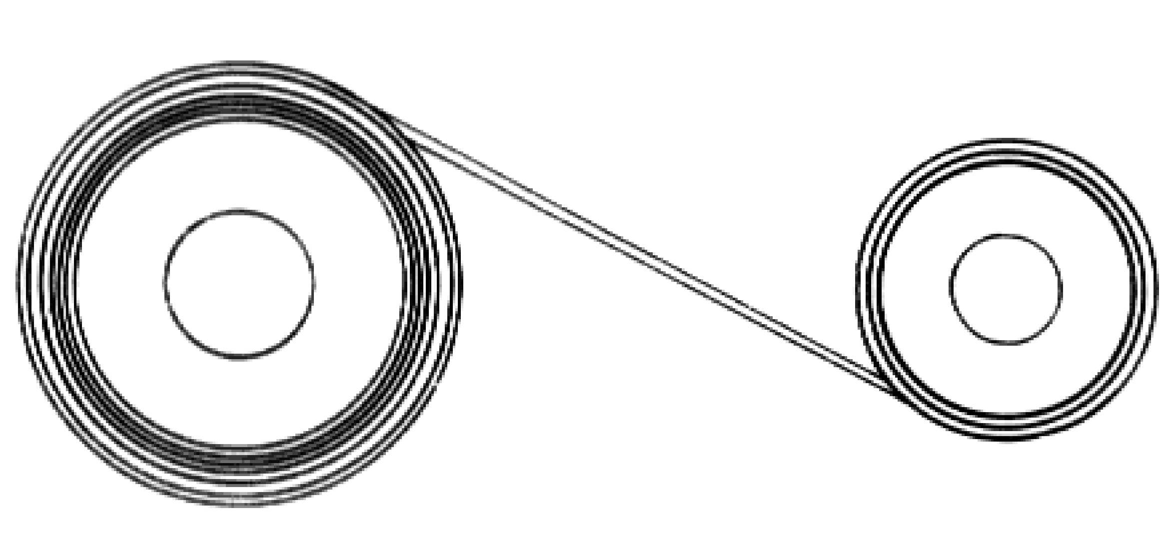

Fig. 45 Motor spring designed for extra power

Whereas it is possible to design constant-torque motors from the elementary principles given, the author has found that the finished spring may be very different from that expected. This is because, unless extreme care is taken, the material bows across its width during the forming operation, which introduces an additional and almost unpredictable variable, causing a degree of instability in the spring that is responsible for a torque–revolution curve that is far from linear.

The suppliers of constant-torque motor springs under their trade names have built up large stocks of standard springs rated in much the same way as electric motors; thus by far the quickest method of obtaining a suitable spring is to specify the requirement and allow the supplier to provide the appropriate spring.

Helical torsion springs

If a close-coiled helical spring is clamped at one end, whilst the other is constrained to rotate about its major axis by an applied torque, the following effects will be observed.

1. If the spring is 'wound up':
   (i) the diameter of the helix decreases,
   (ii) the number of coils increases, and
   (iii) the overall length of the spring increases.
2. If the spring is 'wound down':
   (i) the diameter of the helix increases,
   (ii) the number of coils decreases, and
   (iii) the overall length of the spring decreases.

Care must be taken to ensure that any fittings used in the assembly will provide adequate allowance for the changes in size that occur.

In most cases it is advisable to design torsion springs so that they wind up when the torque is applied, and deliver the required energy before returning to the free position. That is to say that the stress should not be allowed to return to zero during operation, much less change sign by passing through it.

The great majority of torsion springs are manufactured from hard drawn or pre-hardened material, and the action of coiling induces stresses which remain in the wire on release from the coiling mechanism. These stresses are such that the resultant stress is reduced if the spring is subsequently wound up, but increased if it is wound down. If the nature of the application is such that the unwinding mode is essential, then the spring should be given a suitable low-temperature, stress-relieving treatment before use.

This, however, is not the only reason for using the spring in the winding-up mode. Fig. 46 shows a simplified diagram of a torsion spring supported on a central rod and bearing an applied load, *P*, at the end of a lever of length *L*. The maximum stress in the material when the spring is wound up is caused by an applied torque of *PL*, whereas, when the spring is wound down, it is caused by an applied torque of $P\{L + (D/2)\}$, where *D* is the spring diameter.

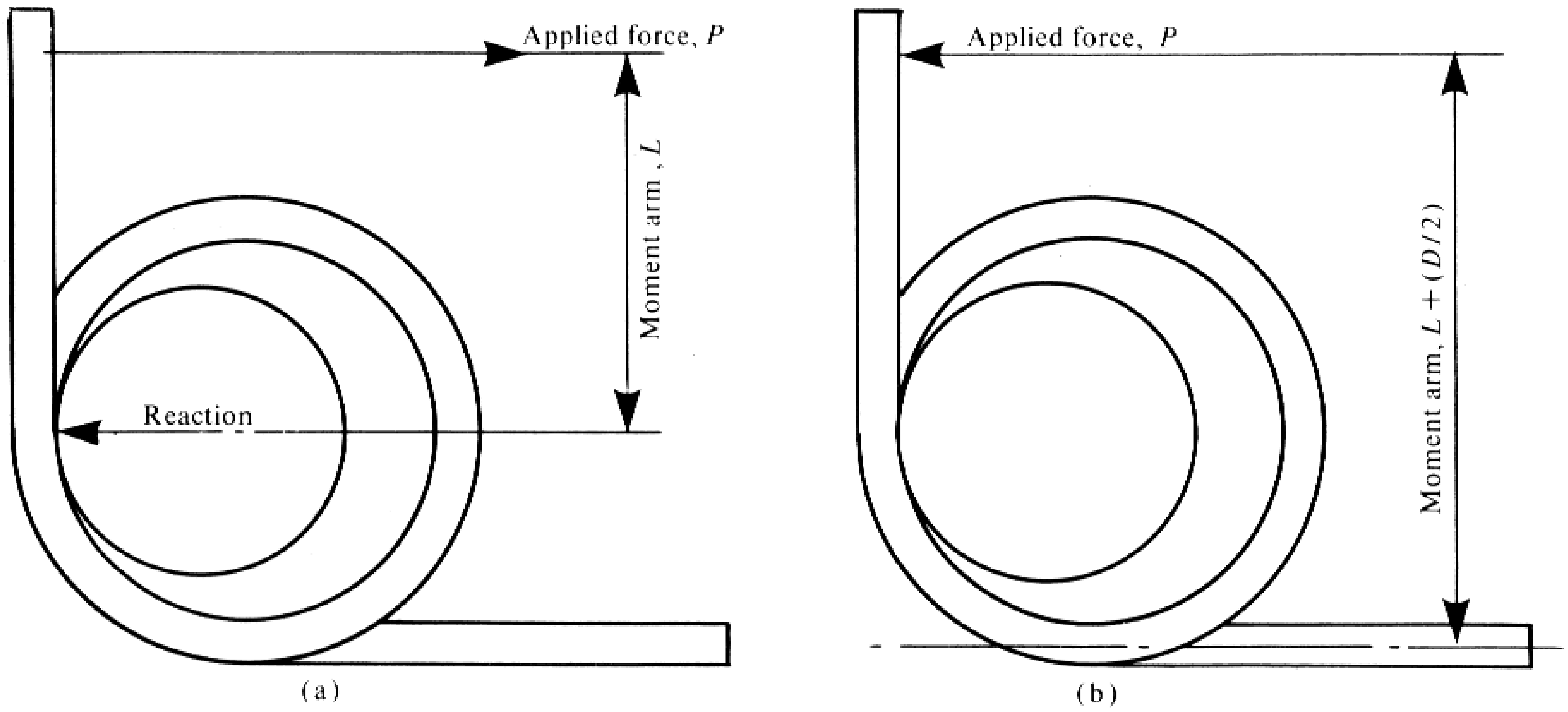

Fig. 46 Comparison of applied torque in (a) winding-up mode and (b) unwinding mode

The fundamental formulae for designing helical torsion springs are relatively simple, being derived from elementary bar-bending theory. The major concern of the designer is to ensure that there is adequate clearance for the spring to operate and to design the ends so that additional movements and increases in stress are properly allowed for.

Round-wire springs

In the case of round-wire springs the stress in the body of the spring is given by

$$f = 10.2M/d^3, \quad (31)$$

where M is torque applied and d is wire diameter.

Sometimes, however, it is necessary to introduce bends of a smaller radius than the body of the spring in order to facilitate fixing the ends, when a correction factor must be used to estimate the higher local stresses at such points. The correction factor, K_s, which is a function of the ratio d/d_c is shown in Fig. 47; d_c is twice the radius of curvature of the bend under consideration.

The angular deflection θ in turns caused by an applied torque M is given by

$$\theta = 10.2Mnd/Ed^4,$$

where n is the number of active coils in the spring. If the body of the spring contains few coils, care should be taken to obtain an accurate estimate of the effect of legs, hooks, etc., as these all form part of the flexible unit of the spring and may be responsible for a significant part of the deflection. If the equation is rewritten

$$\theta = (\pi n D) \times 10.2M/\pi E d^4,$$

it will be seen that the expression within the bracket is the length of wire contained in the body of the spring and it is common practice among designers to write the formula as follows:

$$\theta = 10.2Ml/\pi E d^4, \quad (32)$$

where l is the length of wire in the spring including the effective lengths of wire between the body and the points of application of the load. Reference will be made to this when the various types of fixing in common use are considered (p. 26).

Springs made from material of rectangular section

If the spring is made of material of section bh, where h is the radial depth of the material and b is the width of the section measured parallel to the major axis of the spring, then the stress,

$$f = K_r \times 10.2M/bh^2,$$

where K_r is the stress-correction factor for rectangular-section material as given in Fig. 47. The angular deflection in turns is given by

$$\theta = 6MnD/Ebh^3,$$

which neglects the deflection of the end fixings, or

$$\theta = 6Mnl/\pi Ebh^3,$$

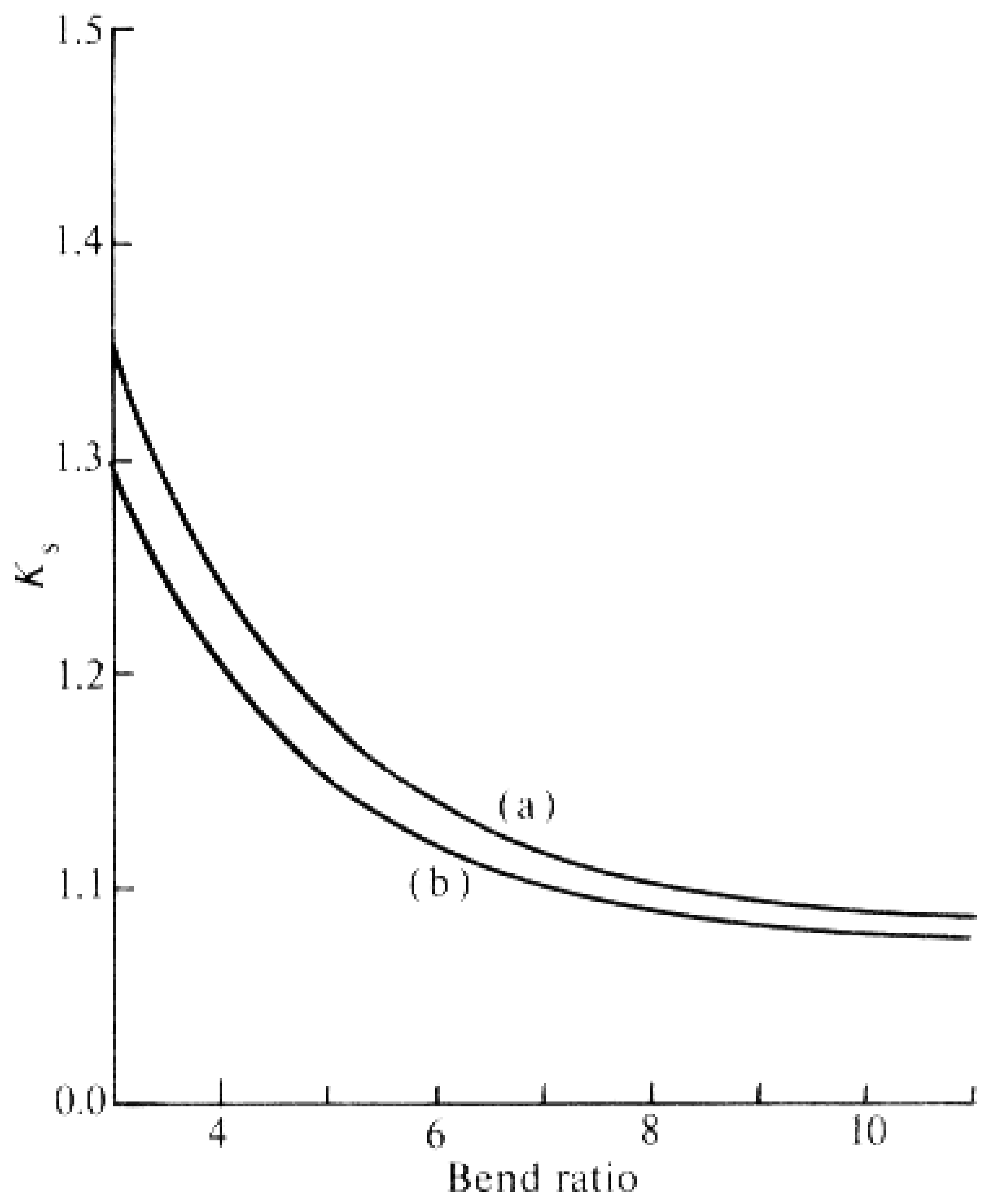

Fig. 47 Stress correction factor, K_s, versus shape ratio: (a) round section (d_c/d); rectangular section (d_c/h)

where l is the estimated total length of material in the spring if the ends are considered to make a significant contribution to the deflection.

Worked example

It is desired to design a helical torsion spring of round-section steel having a maximum outside diameter, D_o, of 9 mm, a minimum inside diameter, D_i, of 4 mm, and a maximum solid length, L, of 14 mm. The maximum torque is to be 1000 N mm with a deflection of 29° and the maximum uncorrected body stress shall not exceed 1200 N/mm². Each end of the spring consists of a straight tangential portion which is considered to contribute 3 mm to its flexibility. The nominal mean diameter,

$$D = \tfrac{1}{2}(9 + 4) = 6.5 \text{ mm},$$

and (from eqn (31)) the maximum stress,

$$f = 1200 = 10.2 \times 1000/d^3 \text{ N/mm}^2$$

Therefore,

$$d^3 = 102,$$

and

$$d = 2.04 \text{ mm}.$$

Selecting the next standard wire size above this (2.12 mm), then (from eqn (32)):

$$\frac{29}{360} = \frac{10.2 \times 1000 \times l}{\pi E d^4};$$

therefore,

$$l = \frac{29\pi \times 207 \times 2.12^4}{360 \times 10.2} = 103.7 \text{ mm}.$$

Since the ends contribute 6 mm of material, the length of wire in the spring is 97.7 mm or 4.8 coils. Since M = 1000 N mm,

$$f = 10.2 \times 1000/2.12^3 = 1070.5 \text{ N/mm}^2.$$

When the spring is wound up the total number of coils,

$$n = 4.8 + 29/360,$$

and the maximum solid length,

$$L = (4.88 + 1) \times 2.12 = 12.47 \text{ mm}.$$

The outside diameter of the spring.

$$D_o = 6.5 + 2.12 = 8.62 \text{ mm}.$$

When the spring is fully wound (4.88 coils) the mean diameter will be 6.5 × 4.8/4.88 = 6.39, so that the minimum inside diameter,

$$D_i = 6.39 - 2.12 = 4.27 \text{ mm}.$$

As the ends are tangential to the body there is no need to apply a stress-correction factor.

Possible adjustments. The design of the mechanism may be such that it is necessary to use a spring in which the number of coils is an exact whole number so that 4.8 will be unsatisfactory. Since the equation for deflection shows that this is dictated solely by the length of wire in the spring the only options open are (1) to vary the size of wire (and this will almost invariably lead to a non-standard size) or (2) to reduce the mean diameter so that

$$5\pi D = 97.7 \text{ mm},$$

whence,

$$D = 6.22 \text{ mm}.$$

When this spring is wound up to full torque the mean diameter will reduce to

$$D = 6.22 \times 5/5.08 = 6.12 \text{ mm},$$

giving a minimum inside diameter of 4.00 mm which, while theoretically acceptable, leaves no room for tolerances, so that it would be inadvisable for the spring manufacturer to attempt to meet such an order. He must now offer the spring user the alternatives of:

(1) modifying the end fixings to vary the active length of wire in the spring, or
(2) spending extra money in order to obtain a material that is non-standard; for example, if a material of diameter 2.14 mm were available then

$$l = \frac{29\pi}{360} \times \frac{207 \times 2.14^4}{10.2} = 107.7 \text{ mm}.$$

Therefore length in body of spring = 101.7 mm, and if n = 5.0, then

$$D = 101.7/5\pi = 6.48 \text{ mm}.$$

Thus free outside diameter = 8.62 mm, free inside diameter = 4.34 mm, mean diameter when wound up = 6.48 × 5/5.08 = 6.38 mm, minimum inside diameter = 4.24 mm, and the spring is now satisfactory.

Springs of non-circular section material

The availability of round-section wire and the ease with which it may be coiled and formed into torsion springs ensure that it is used for the manufacture of most springs of this type. If, however, it is required to store the maximum energy in a limited space, it may be advisable to use a square or rectangular section. The uneven distribution of stress caused by the curvature of the material locates the highest stress at its inside edge, which has a much greater area than in a wire or circular section that in theory represents only a single point of contact.

A further advantage of the rectangular section is that considerable adjustments may be made to the overall length and diameter of the spring by minor adjustments of the thickness or breadth of the material, or both. However, not only is rectangular material difficult to obtain and its general quality inferior to that of round section, but also it distorts during coiling and becomes trapezoidal with the greatest growth on the inside, so that much of the available space in the spring is unused. This may be overcome by starting with a trapezoidal section and coiling it 'inside-out' so that in theory it becomes rectangular. It is possible to calculate the original shape from the properties of the material but it must be realized that this shape will apply only to the spring for which it has been designed and will be unsatisfactory for other work. Unless the order warrants the purchase of the whole output of that particular shape, the material will not be available.

Fig. 48 shows four different types of commonly used symmetrical ends. It is, of course, not necessary that the ends should be symmetrical and any shape may be dictated by the user.

Also illustrated in Fig. 48 is a double torsion spring in which the two ends are wound to opposite hands joined by a 'bridge' which may be of any shape. Such springs are normally used in pairs, as shown in Fig. 49, where it is necessary to exert a load along a constant line of action. Probably the most common duplex torsion springs are those used on motor-cycle

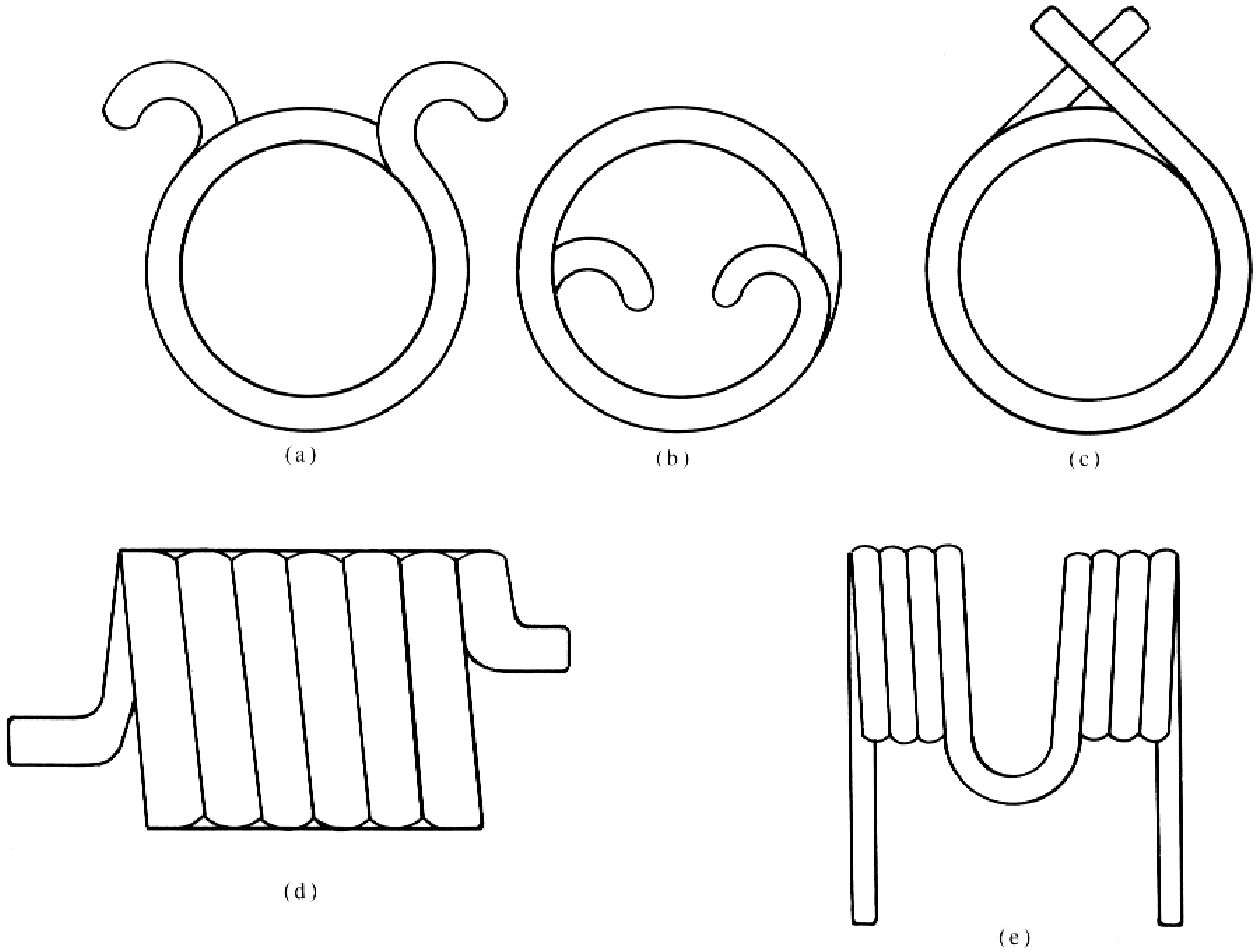

Fig. 48 Torsion spring ends: (a) external hooks, (b) internal hooks, (c) tangential arms, (d) offset arms, (e) double torsion

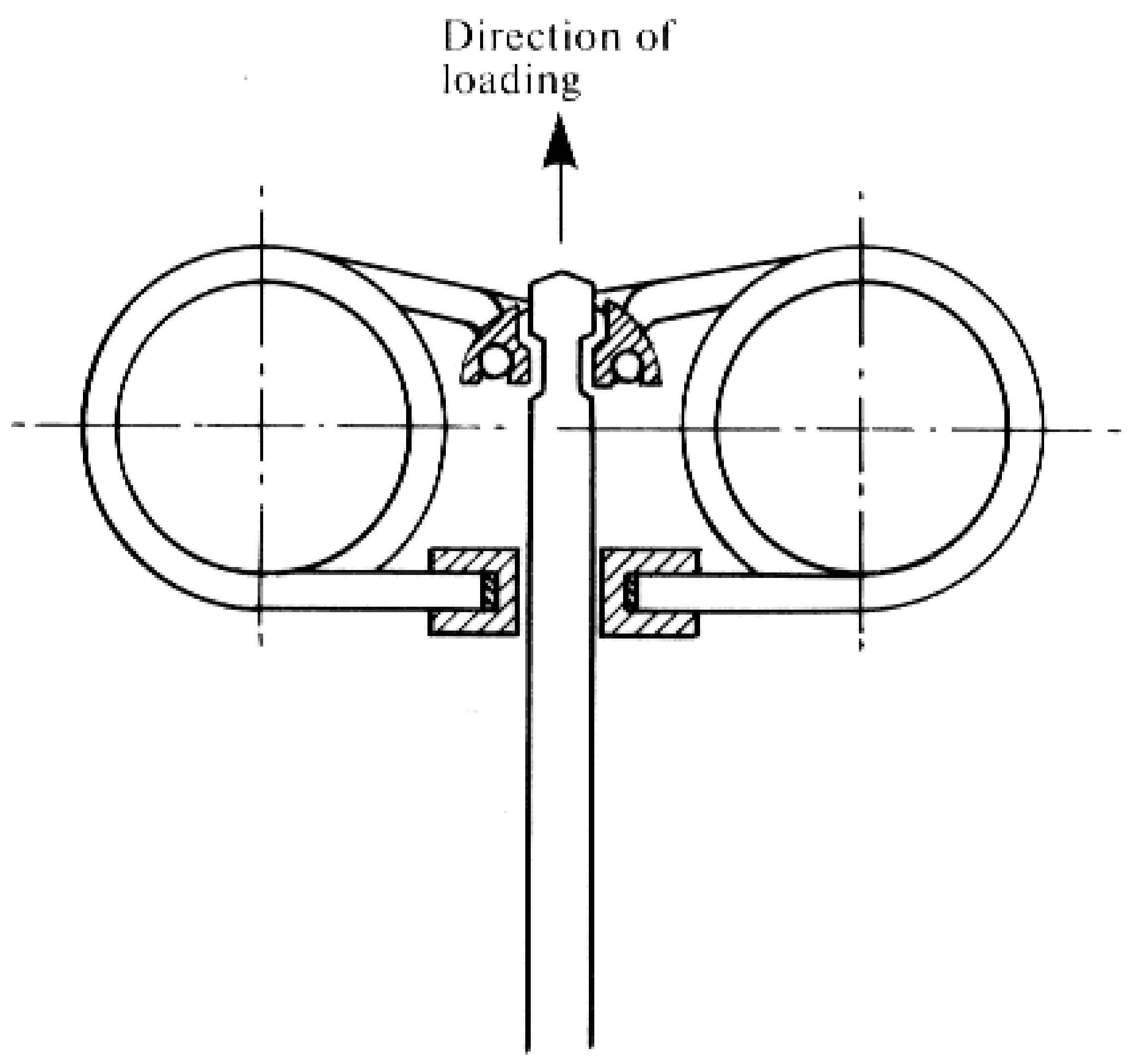

Fig. 49 Schematic arrangement of duplex torsion spring assembly

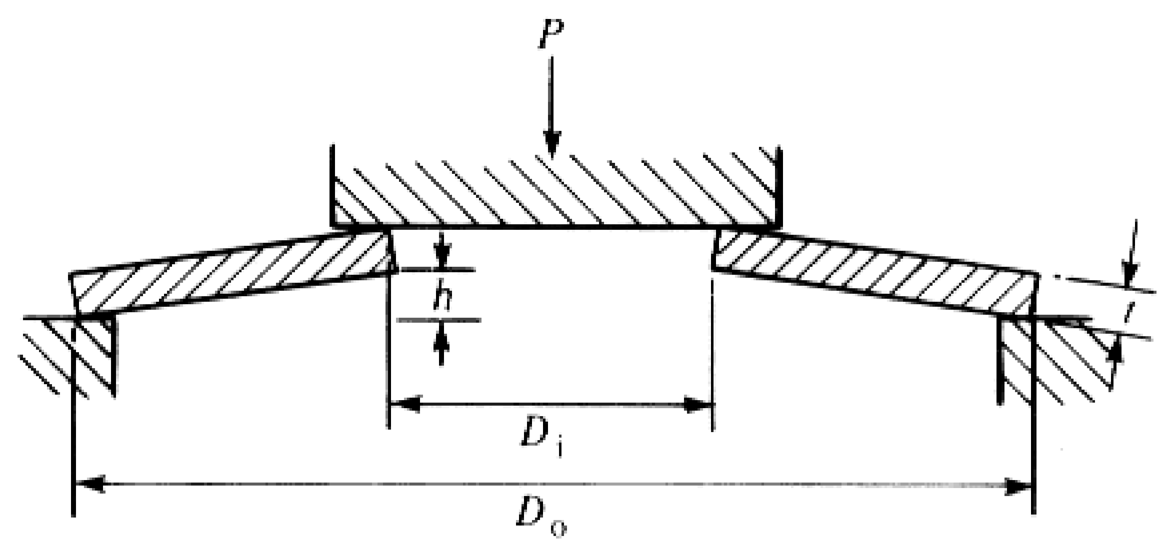

Fig. 50 Initially coned disc spring

engines. The loads to be applied are divided into four and the end-section of each spring is designed as a single torsion spring to carry one fourth part of the load. The thrust is usually taken on straight legs which must be free to slide as the body moves up and down, while the load is applied through a saddle designed to accommodate the ends.

Initially coned disc springs

An initially coned disc spring—often referred to as a 'Belleville washer', although this term should strictly only be applied to a disc having one particular set of dimensional ratios as specified in the original patent—consists of an annular disc of constant thickness raised to the form of a truncated cone as illustrated in Fig. 50. In this figure D_o is outside diameter measured to the centre of the outer edge, D_i is inside diameter measured to the centre of the inner edge, t is material thickness, and h is coned height.

The load–deflection characteristic of a disc of this type is a multinominal function of inside diameter, outside diameter, material thickness and coned height, and many differently shaped load–deflection curves may be obtained by varying the ratio of h/t. Three typical curves having $h/t = 3.0, 1.5$, and 0.6 are shown in Fig. 51, and although it must be stressed that these are not drawn to the same scale they illustrate the wide variety in form that can be produced.

When $h/t = \sqrt{2}$, the tangent to the curve remains horizontal for a considerable distance, and although

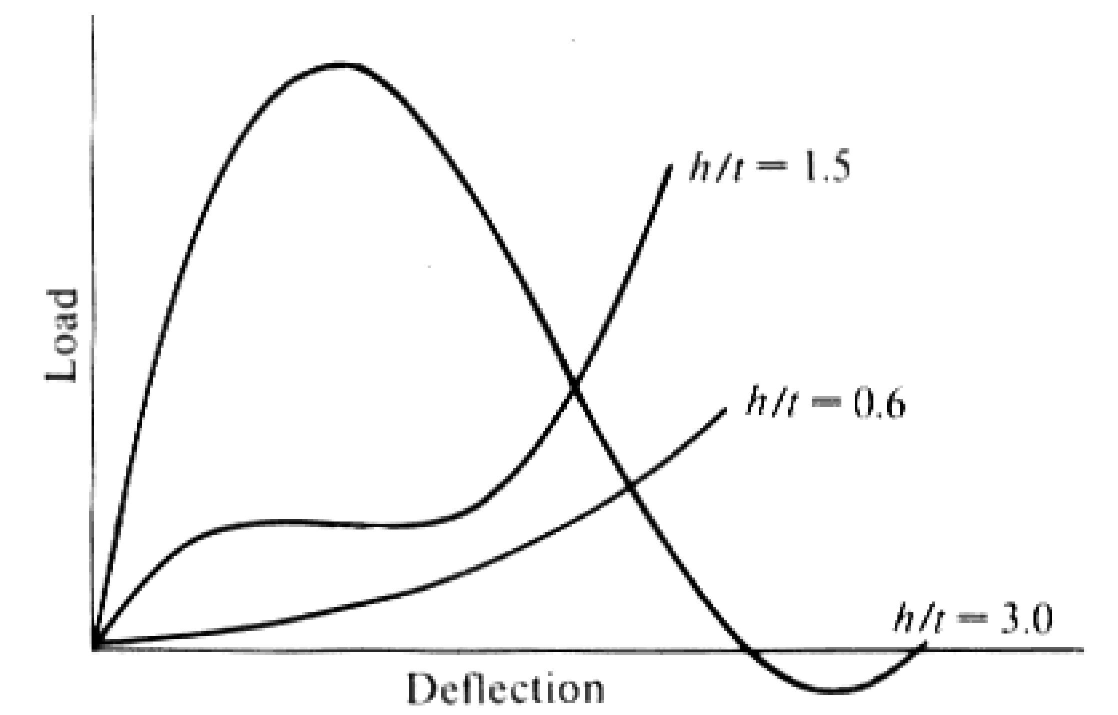

Fig. 51 Typical load–deflection curves

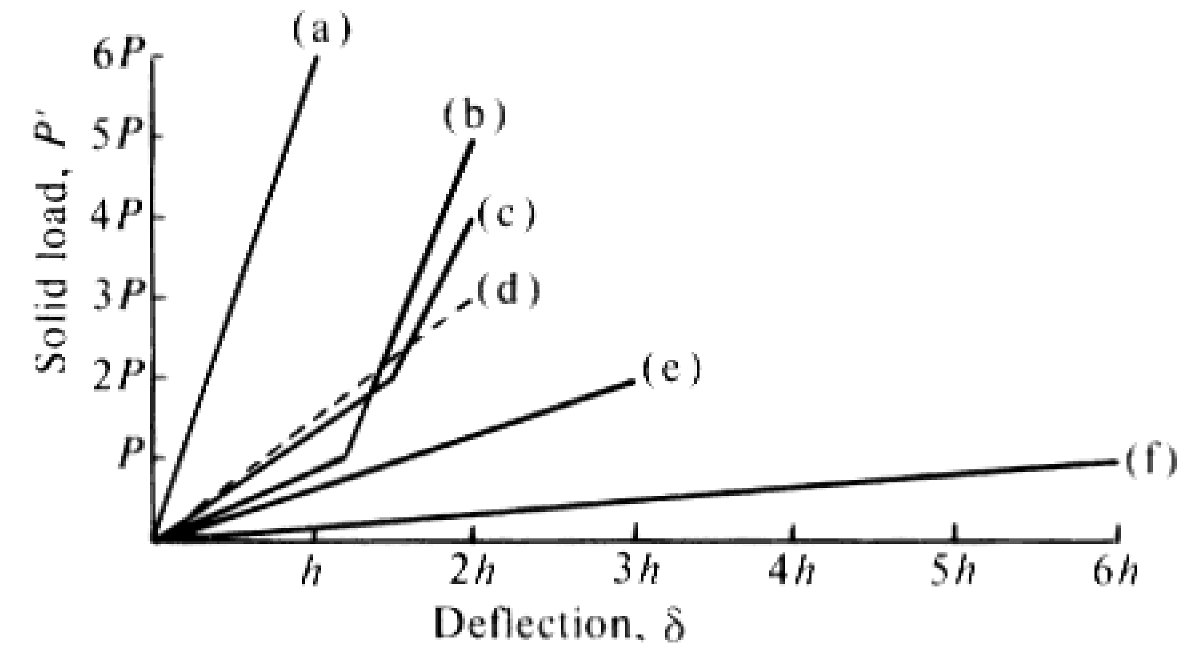

Fig. 53 Load–deflection characteristics for arrangements of the six linear discs shown in Fig. 52

h/t = 1.5 produces a slight peak and subsequent fall before rising again it is this value that produces the longest zone of very low rate.

Further variations in load–deflection characteristics may be produced by stacking the discs in series or parallel; six of the many possible arrangements using six discs are illustrated in Fig. 52, while Fig. 53 shows the load–deflection characteristics of these arrangements assuming that the discs are designed to be individually linear.

As reference to Fig. 51 shows, when designing discs to be stacked in series care must be taken to see that the h/t ratio does not approach 1.5 since this leads to a danger of individual discs snapping inside out and producing a stack of very different characteristics.

Those handling discs with ratios in excess of 1.5 should be warned that since these springs have been flattened during test they may assume an inside-out configuration, and although they are virtually identical to 'normal' discs they are now reservoirs of stored energy requiring only a sharp blow to cause them to snap back to their original form with an almost explosive energy release, possibly causing the incautious to receive severe finger damage.

Deflection

In order to reduce the difficulty of analysing the load–deflection characteristic of a disc spring by the application of elastic theory it is now accepted practice to assume that the radial cross-section of the disc rotates about its point of support without distortion and, provided that the diameter ratio D_o/D_i is not much greater than three, the error so introduced does not exceed 5 per cent.

Making this assumption Almen and Laszlo (1936) arrived at the following relationship:

$$P = \frac{4E\delta}{(1 - \mu^2)\, D_o^2} \times [C_2(h - \delta)\{h - (\delta/2)\}t + C_2''t^3],$$

where P is axial load, δ is axial deflection, μ is Poissons ratio,

$$C_2 = \pi \left(\frac{\alpha + 1}{\alpha - 1} - \frac{2}{\log_e \alpha}\right) \left(\frac{\alpha}{\alpha - 1}\right)^2,$$

and

$$C_2'' = \pi \log_e \alpha \times \{\alpha/(\alpha - 1)\}^2/6,$$

where $\alpha = D_o/D_i$.

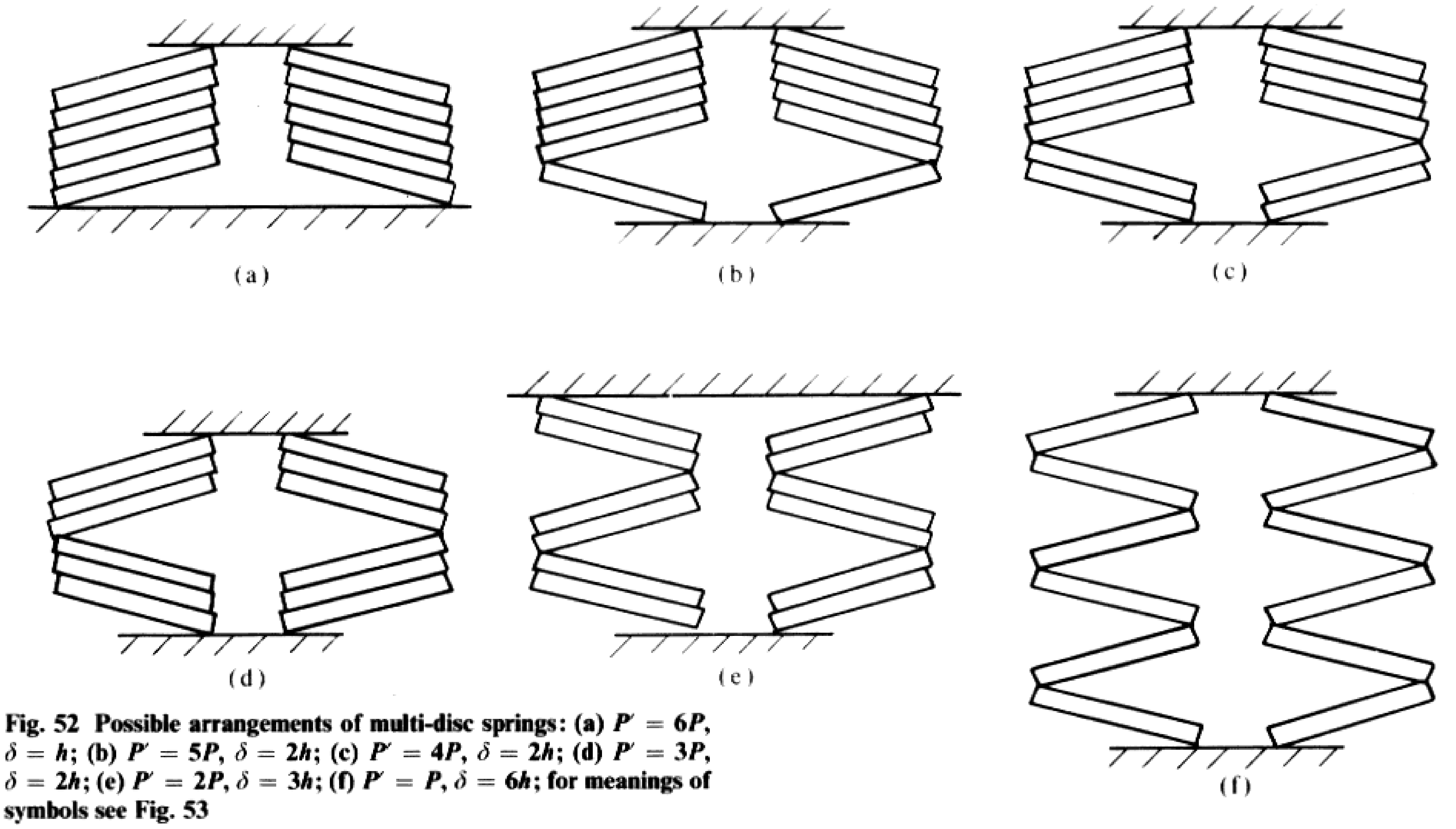

Fig. 52 Possible arrangements of multi-disc springs: (a) $P' = 6P$, $\delta = h$; (b) $P' = 5P$, $\delta = 2h$; (c) $P' = 4P$, $\delta = 2h$; (d) $P' = 3P$, $\delta = 2h$; (e) $P' = 2P$, $\delta = 3h$; (f) $P' = P$, $\delta = 6h$; for meanings of symbols see Fig. 53

A relatively simple calculation shows that within practical limits $C_2 = C_2''$ so that a simplified expression for load in terms of deflection is

$$P = \frac{4E\delta C_2}{(1-\mu^2)D_o^2} \times [(h-\delta)\{h-(\delta/2)\}t + t^3]. \quad (33)$$

To simplify design computation, values of C_2 as a function of diameter ratio α are shown as a curve in Fig. 54, and using this curve and rewriting eqn (33),

$$P = 4Et^4C_1C_2/D_o^2, \quad (34)$$

where

$$C_1 = \frac{\delta}{(1-\mu^2)t}\left\{\left(\frac{h}{t}-\frac{\delta}{t}\right)\left(\frac{h}{t}-\frac{\delta}{2t}\right)+1\right\}. \quad (35)$$

Factor C_1 may be plotted as a function of δ/t and h/t. Values of C_1 for h/t ratios between 0.4 and 3.0 may be obtained from Figs 55–58. It must, however, be noted that when $\delta = h$ the washer has been pressed flat and unless special arrangements in fitting have been made, no further deflection is possible. In Figs 57 and 58 the points corresponding to $\delta = h$ or $\delta/t = h/t$ have been marked on each curve and the points joined by a dotted line, and unless the assembly allows the washers to be pushed 'through' or beyond flat only that portion of the diagram to the left of this line is available for design purposes. The special case $h/t = 1.5$ is shown on an enlarged scale in Fig. 59.

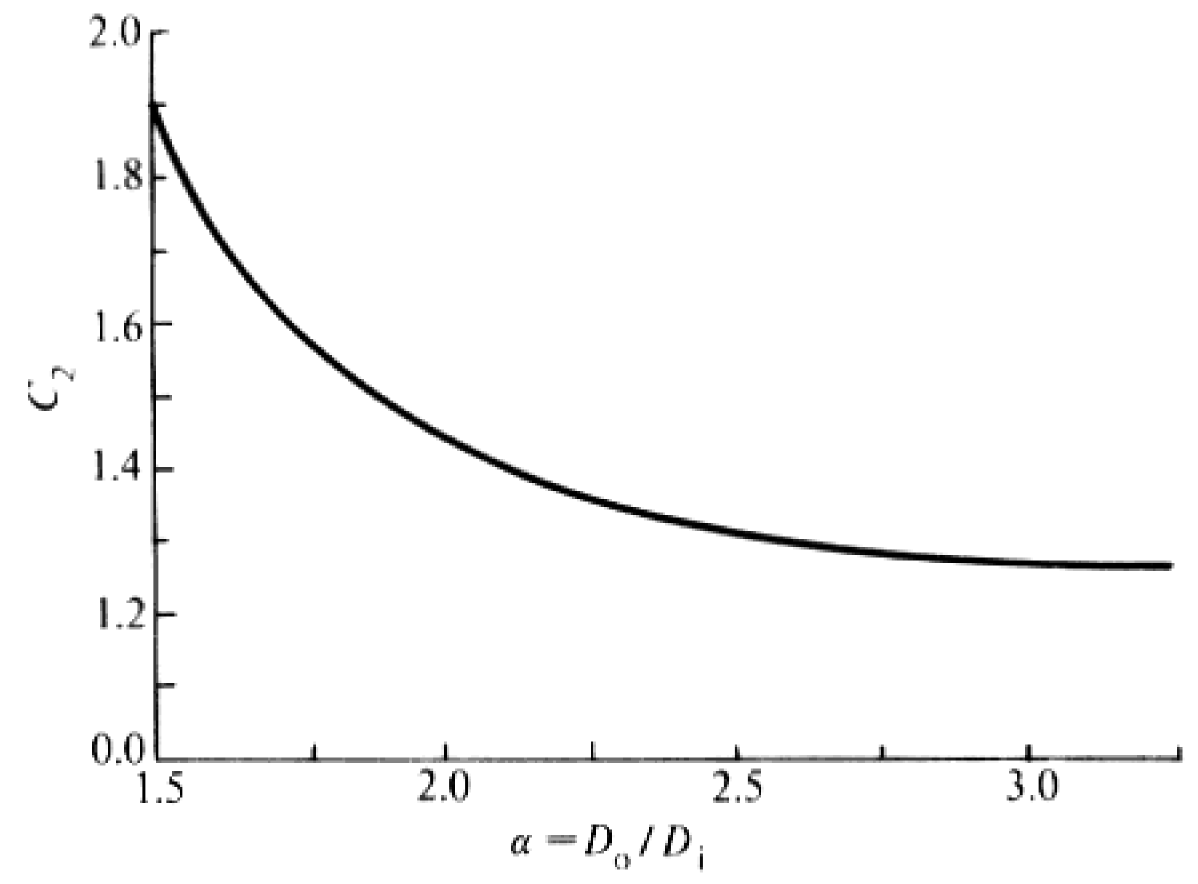

Fig. 54 Values of design factor, C_2; expression for C_2 in terms of α given in the text

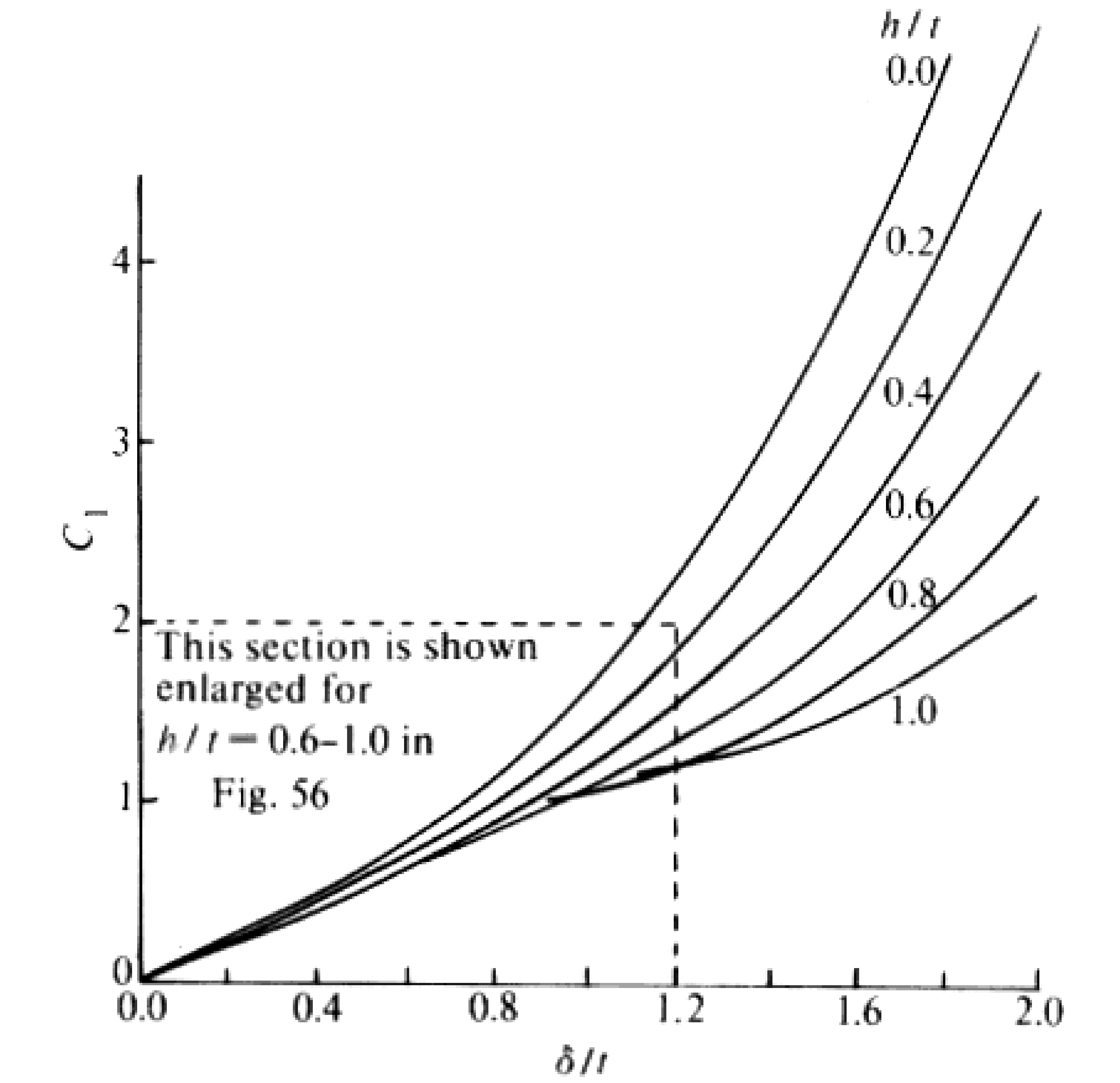

Fig. 55 Values of design factor, C_1 (see eqn 34)

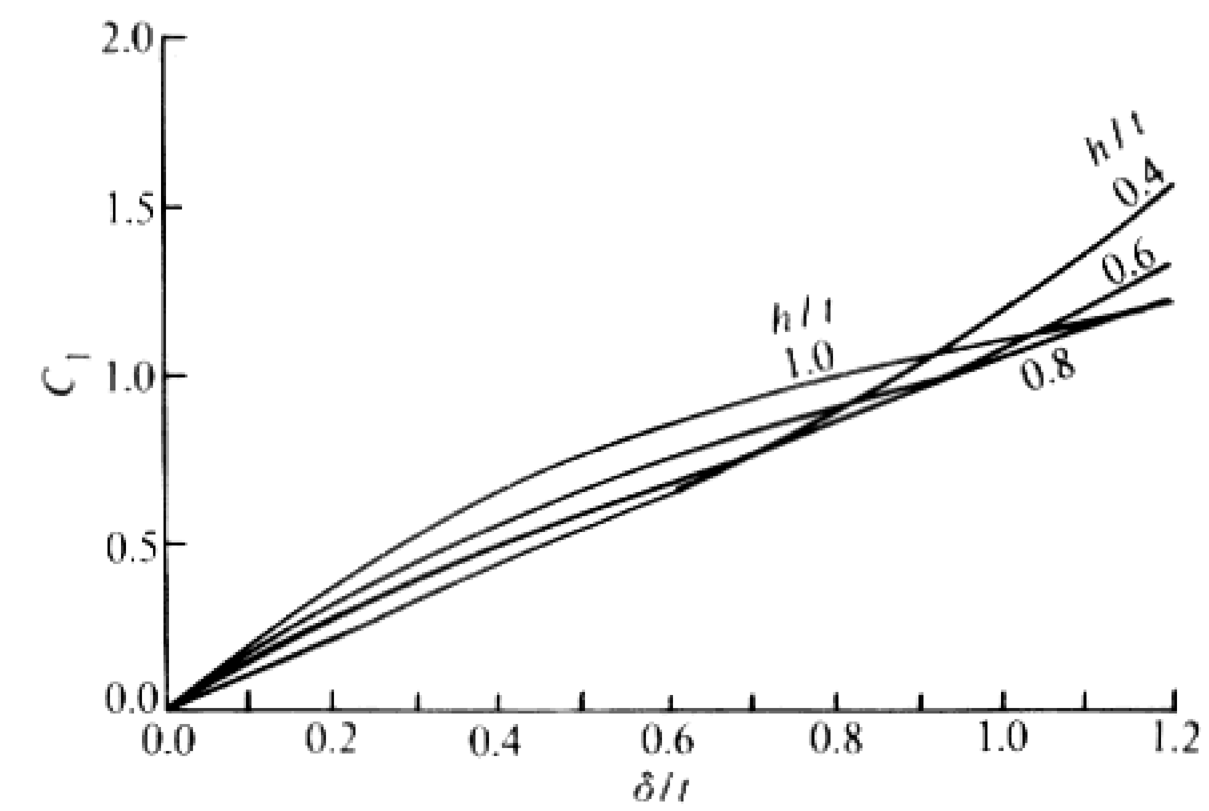

Fig. 56 Graph showing enlarged section from Fig. 55 for h/t = 0.4–1.0

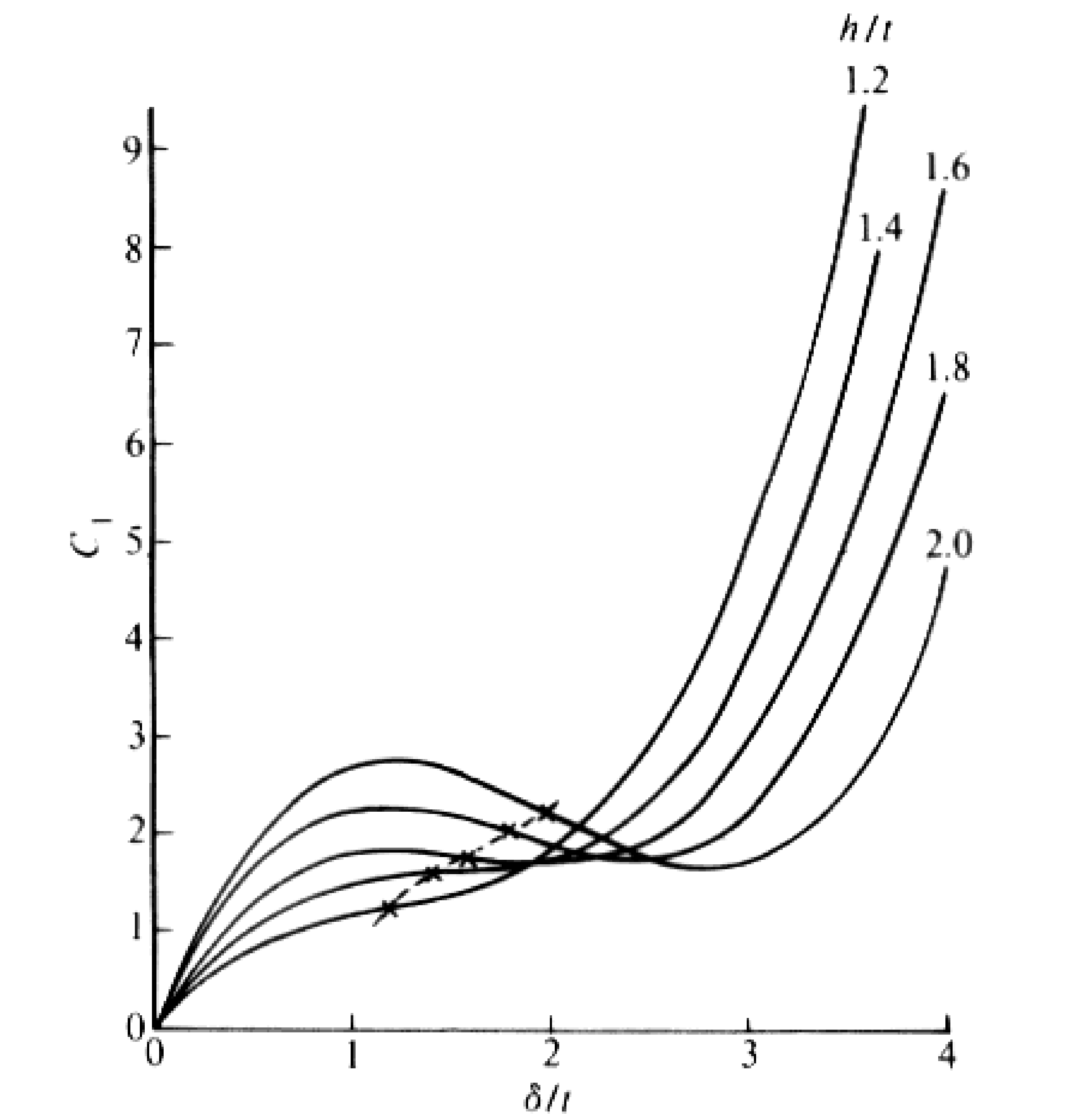

Fig. 57 Values of C_1 for values of h/t = 1.2–2.0

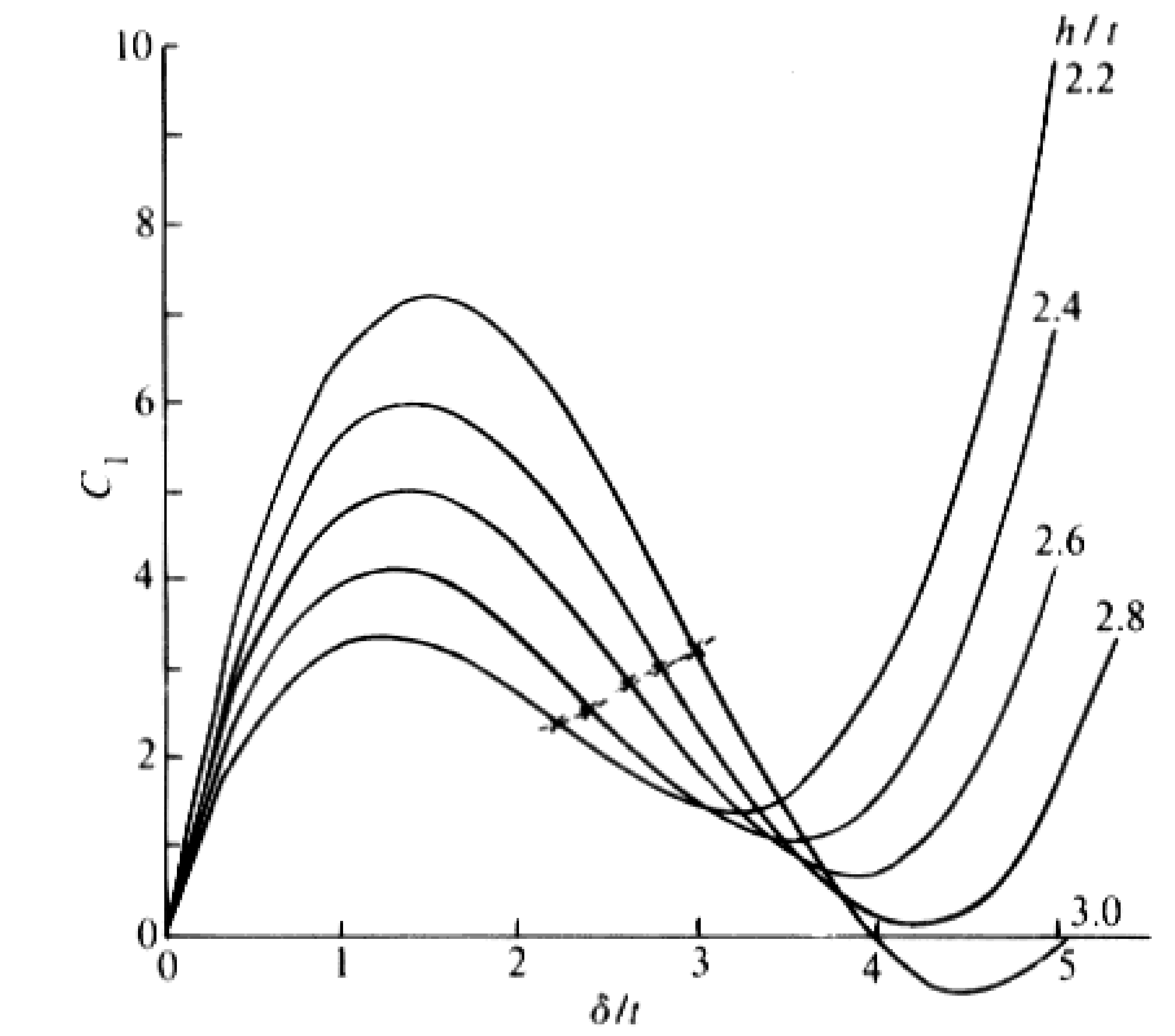

Fig. 58 Values of C_1 for values of h/t = 2.2–3.0

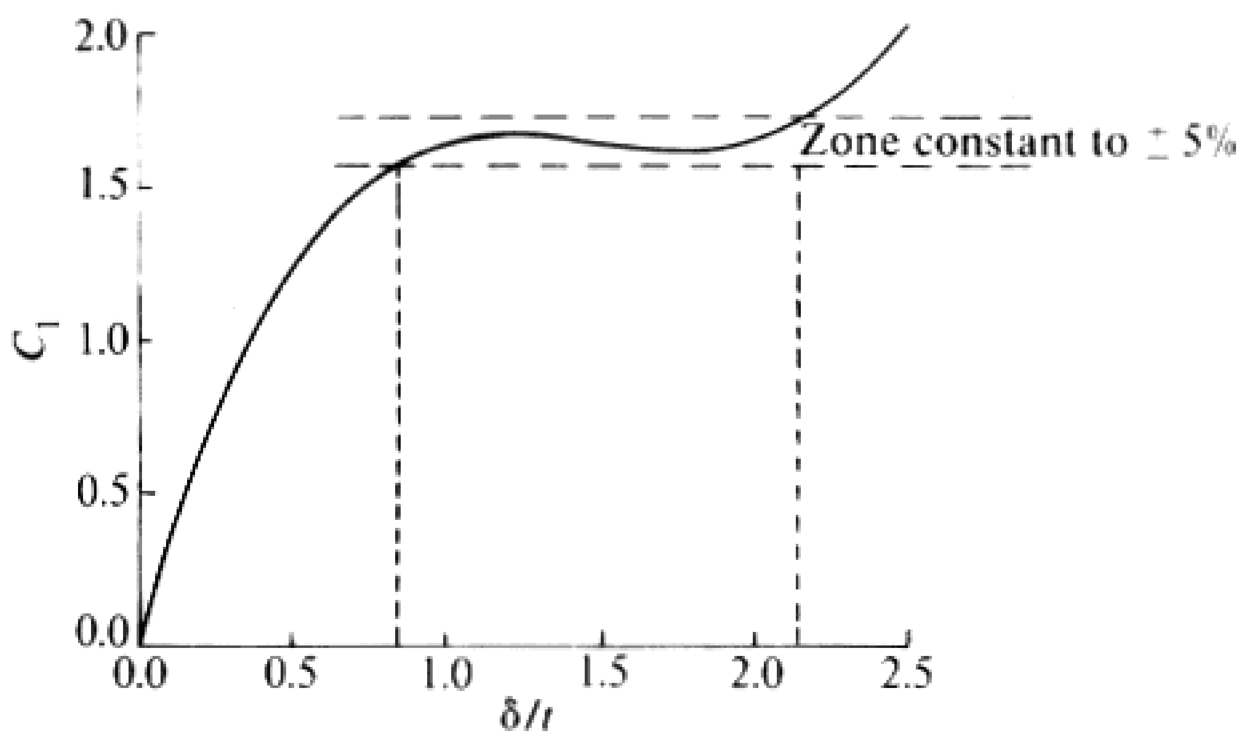

Fig. 59 Deflection characteristic of constant-load spring (h/t = 1.5)

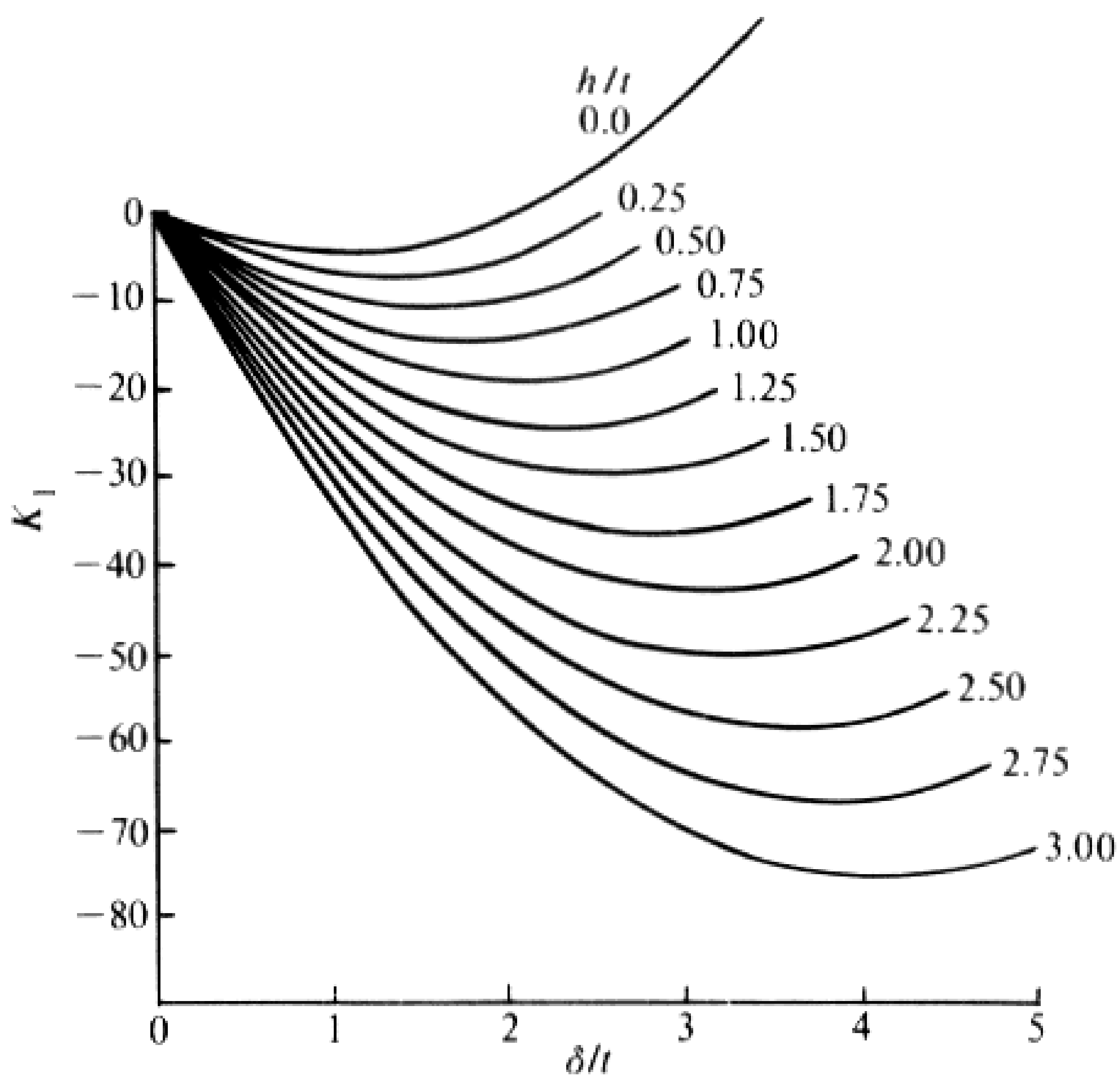

Fig. 60 Stress factor K_1 for D_o/D_i = 1.5

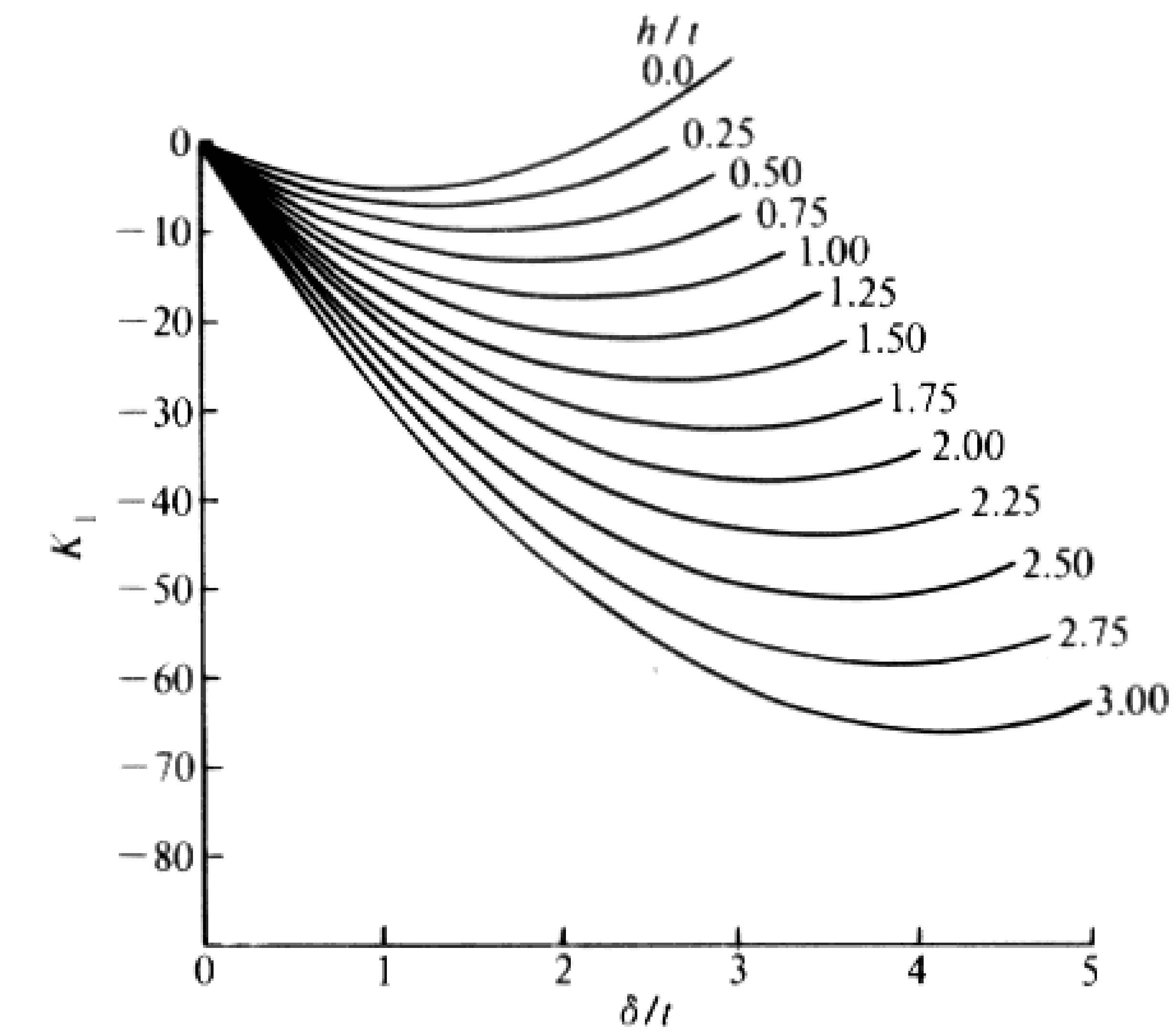

Fig. 61 K_1-values for D_o/D_i = 2.0–2.5

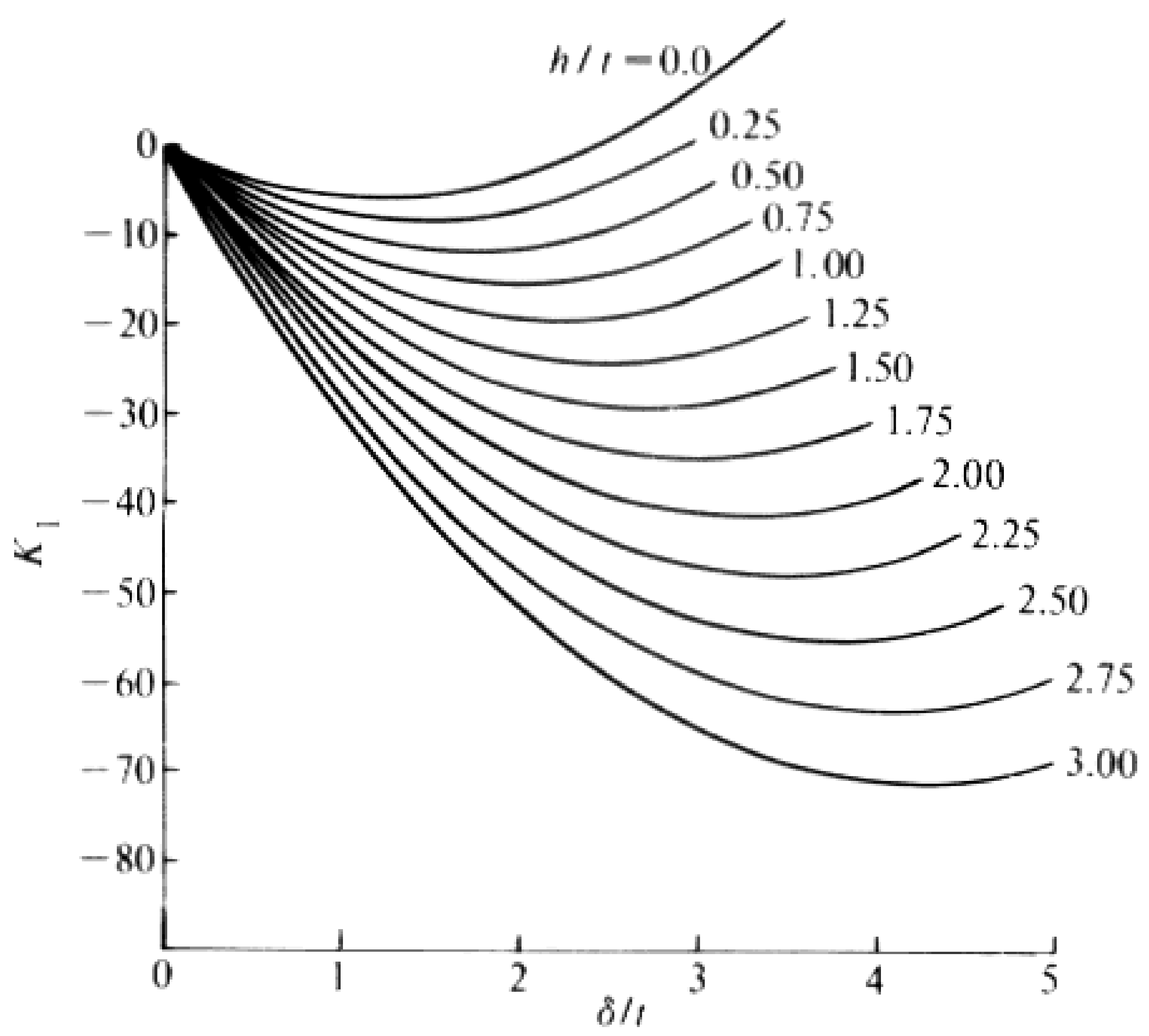

Fig. 62 K_1-values for D_o/D_i = 3.0

Stress

It is just as tedious and complicated to analyse stress from first principles as it is deflection. However, by making the same assumptions the maximum stress at the upper surface of the spring may be expressed as

$$f_u = \frac{-4E\delta C_2}{(1-\mu^2)D_o^2} \times [C_3\{h-(\delta/2)\} + C_4 t], \quad (36)$$

and that at the lower inner edge as

$$f_i = \frac{-4E\delta C_2}{(1-\mu^2)D_o^2} \times [C_3\{h-(\delta/2)\} - C_4 t], \quad (37)$$

where $C_3 = \dfrac{6}{\pi \log_e \alpha} \times \left(\dfrac{\alpha - 1}{\log_e \alpha}\right)$

and $C_4 = \dfrac{3(\alpha - 1)}{\pi \log_e \alpha}$.

Negative values for f_u and f_i signify that the stress is compressive, whereas positive ones indicate that it is tensile.

In order to simplify the calculation of stress, eqn (36) may be re-written:

$$f_u = K_1 E t^2 / D_o^2, \quad (38)$$

where

$$K_1 = \frac{-4\delta}{t} \times \frac{C_2}{(1-\mu^2)} \left\{ C_3\left(\frac{h}{t} - \frac{\delta}{2t}\right) + C_4 \right\}, \quad (39)$$

and since eqn (37) only differs from eqn (36) by the sign preceding C_4, then

$$f_i = K_2 E t^2 / D_o^2,$$

where

$$K_2 = \frac{-4\delta}{t} \times \frac{C_2}{(1-\mu^2)} \left\{ C_3\left(\frac{h}{t} - \frac{\delta}{2t}\right) - C_4 \right\}.$$

Figs 60, 61 and 62 show values of K_1 for α = 1.5, 2 to 2.5, and 3. Since the families of curves for $\alpha = 2$ and $\alpha = 2.5$ are almost identical, the differences, drawn to this scale, are indistinguishable. It would of course, be possible to produce a similar set of curves for the factor K for the calculation of the stress in the lower inner edge, but since this is always less by $(4\delta/t)$ $(C_2/1-\mu^2)\ 2C_4$ than the stress in the upper edge, which most of the calculations will have been carried out to obtain, such additional curves are of little benefit.

Simple design

The main area for the use of simple coned-disc design occurs where there is a severe space restriction in the direction of application of the load while the deflection is of the same order as the 'solid' height space available. Under such conditions the dead or inactive coils of a compression spring would occupy much of its vertical height and preclude its use.

Worked example

A disc spring is required to carry a load of 10 000 N with a deflection of 7 mm and a possible 10 per cent overrun before it becomes solid. The solid height is not to exceed 8.25 mm, the outside diameter is restricted to 180 mm, and the maximum stress to 1450 N/mm².

(1) To calculate the height ratio.
Total movement required = 7 + 0.7 = 7.7 mm.
Solid height ≈ 8.25 mm.
Height ratio ≈ 7.7/8.25 = 0.93.

(2) To calculate the diameter ratio. This usually lies between 1 and 3 and it is customary to select a value at random and adjust it later if necessary. In the present example a value is taken that is near the middle of the range and is as simple as possible (say $\alpha = 2$).

(3) Stress. In Fig. 61 take $h/t = 1.0$ and $\delta/t = 1.0$, which represents the condition of a washer pressed flat and gives the value of K_1 for maximum compressive stress in the top inner edge. In this case $K_1 = -13$. (The negative sign indicates a compressive stress and will be disregarded in the simple calculations as it is necessary to take a square root.) Now

$$f_u = K_1 E t^2 / D_o^2$$

$$t = (f_u D_o^2 / K_1 E)^{\frac{1}{2}}.$$

It is of course necessary to ensure that E, P, and f are all expressed in the same system of units. In this case take E = 208 000 N/mm² and F = 1450 N/mm², whence

$$t = \{1450 \times 175^2/(13 \times 208\,000)\}^{\frac{1}{2}} = 4.05 \text{ mm},$$

which is the *maximum* thickness that will permit the deflection without exceeding the size that will be readily available (4 mm).

The design is now as follows: material thickness = 4 mm, inside diameter = 87.5 mm, outside diameter = 175 mm, and coned height = 4 mm.

(4) Check. In Fig. 56 take $h/t = 1$, $\delta/t = 1$, giving $C_1 = 1.1$. Then,

$$\text{solid load} = \frac{4 \times 208\,000 \times 4^4 \times 1.1 \times 1.45}{175 \times 175} = 11\,093 \text{ N},$$

and

$$\text{solid stress} = \frac{-13 \times 208\,000 \times 16}{175 \times 175} = -1413 \text{ N/mm}^2.$$

(5) Number in pack. Solid height = 8.25 mm (max.) and the solid height of a single washer = 4 mm; thus the number in pack = 2, giving a final design of 2 washers in series as previously specified.

Specialist designs

1. Constant load or zero-rate. It is very often necessary to spring load an engineering device in such a manner that differential expansion of the spring housing, although producing considerable variation in fitted length, does not produce a corresponding variation in the spring load. In most devices of this kind the load-carrying member passes through the centre of the spring and restrictions are thus imposed on the designer, whereas the height in the axis of the spring must be kept to a minimum. In these circumstances the zero-rate ($h/t = \sqrt{2}$) disc spring would seem to be the most suitable, in spite of the flat portion of the $h/t = 1.5$ curve (see Fig. 57) being longer (even if not strictly horizontal).

Where such a spring is required, the amount of design work can be considerably reduced by making use of the observation that when $\delta = h$ the spring is approximately halfway along the horizontal portion of its curve. If a maximum stress is selected and strictly adhered to, then the simple curve shown in Fig. 63 linking stress, diameter ratio, and the ratio D_o/t, will enable t to be easily found. Considerations of space will lead the designer to choose suitable inside and outside diameters from which to start, but it must be stressed that if the design fails to meet the specification a new start must be made. It is not sufficient to change one of the variables since this will result in considerable modification of the spring characteristic, whose preservation is the designer's primary aim.

Worked example

A disc spring having the characteristic for $h/t = 1.5$ is required to maintain a constant load of 1050 N ± 5 per cent. Space restrictions dictate an inside diameter of 50 mm and a maximum outside diameter of 130 mm. Stress at constant load is to be approximately 1250 N/mm².

Because of space limitations select $D_i = 50$ mm, $D_o = 125$ mm and $\alpha = 2.5$. Then from Fig. 57,

$$C_1 = 1.70,\ C_2 = 1.32, \text{ and } D_o/t = 69.5,$$

whence $t = 1.8$ mm.

Re-writing eqn (34) as

$$P = 4EC_1C_2(t/D_o)^2 t^2,$$

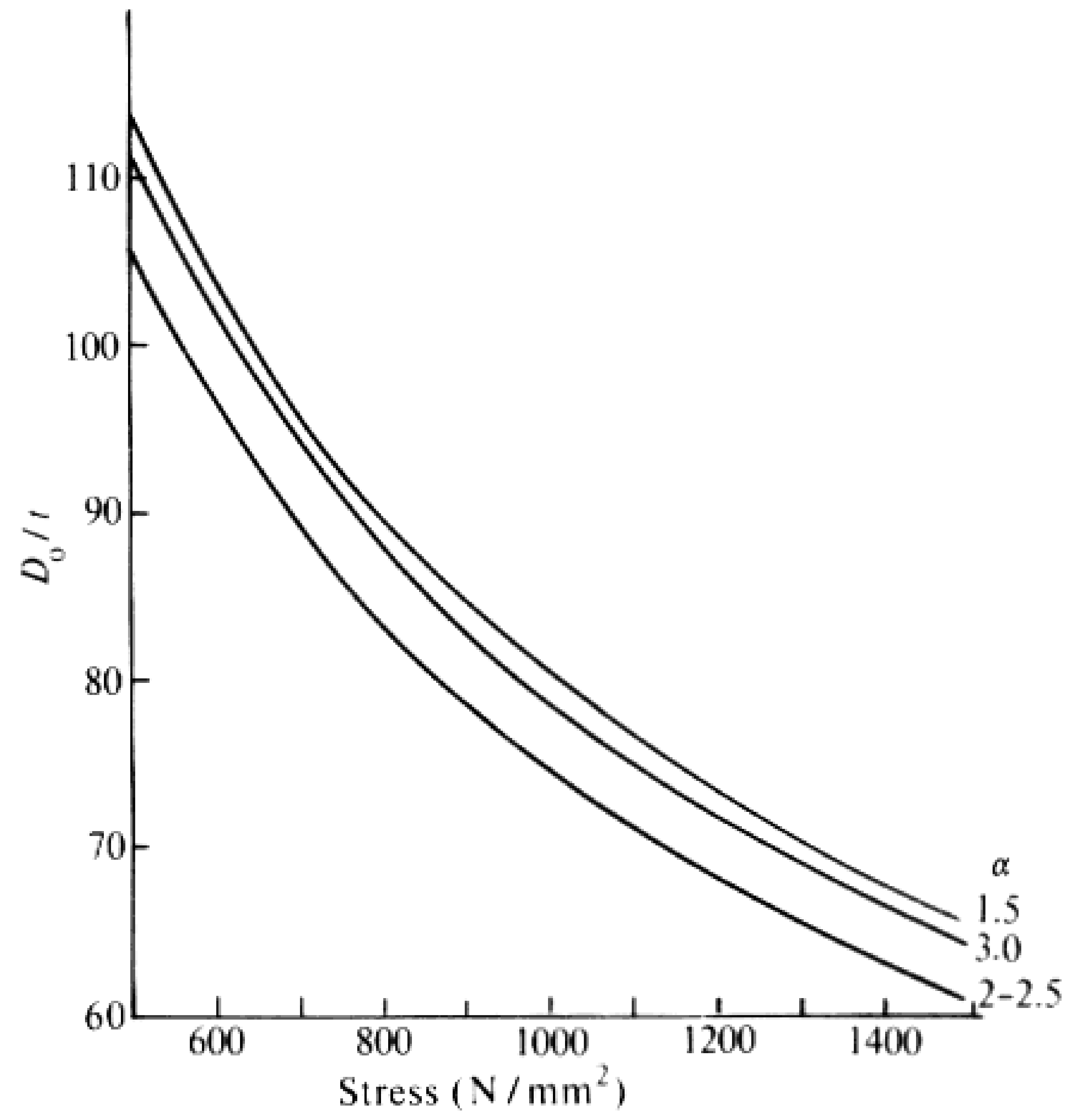

Fig. 63 Design ratio D_o/t for constant load at a given stress ($\alpha = D_o/D_i$)

then

$$P = \frac{4 \times 208\,000 \times 1.7 \times 1.32 \times 1.8^2}{69.5^2} = 1252 \text{ N}.$$

This value is too high and it is not sufficient merely to decrease the material thickness since all the other variables must change to maintain the '1.5' characteristic.

Selecting a smaller diameter (say 115 mm), then $D_i = 50$ mm, $D_o = 115$ mm, and $\alpha = 2.3$.

Since the curves for $\alpha = 2.0$ and $\alpha = 2.5$ are almost identical, then the same constants are retained and $t = 1.65$ mm. Hence

$$P = \frac{4 \times 208\,000 \times 1.7 \times 1.32 \times 1.65^2}{69.5^2} = 1052 \text{ N}.$$

The final design is therefore: material thickness = 1.65 mm, $D_i = 50$ mm, $D_o = 115$ mm and coned height = 2.48 mm.

Such a design would, of course, necessitate the use of ground stock, controlled heat-treatment, careful setting to the correct camber, etc., and would be extremely expensive unless the quantity requirement were large.

2 Snap-action disc springs. On occasion it is necessary to utilize a spring where the characteristic is given by $h/t \geqslant 2.5$. The main purpose of this is to supply a sufficient supporting force until a specified distance has been covered by the mechanism, and then to produce a 'snap' action in which most of the load is almost instantaneously removed. Some of the many applications include 'fail-safe' overload switches, security devices, and 'overcentre' clamps where it is usual for the engineer to specify the shape of the curve required, the maximum load, and the space limitations.

Worked example

A spring characteristic approximating to the $h/t = 3.0$ curve is required in which the maximum load to be supported is 3800 N, after which the load is to fall away rapidly until the spring meets a stop restricting the total movement to 7 mm, representing a point on the curve of $\delta = 3.5t$. Maximum stress is to be kept below 1450 N/mm².

Since $\delta = 3.5t = 7$ mm, then $t = 2$ mm.

Since $h/t = 3$, then $h = 6$ mm.

Starting with the customary shape ratio $\alpha = 2$, take from Fig. 61 $h/t = 3$ and $\delta/t = 3\frac{1}{2}$ to give $K_1 = 64.1$, ignoring the negative sign since only the magnitude of the stress is significant in this context. From, eqn (38)

$$D_o^2 = K_1 E t^2 / f_u.$$

and assuming a maximum stress of 1400 N/mm²

$$D_o = 192 \text{ mm}.$$

From eqn (34) the load corresponding to the peak value of the curve for $h/t = 3$ in Fig. 58 is 7394 N. Thus the load in the fully-deflected position, where $\delta/t = 3.5$, is given by

$$7394 \times C_{1\,3.5}/C_{1\,\max}$$
$$= 7394 \times 1.39/7.2$$
$$= 732 \text{ N}.$$

Applications and materials

Since the peculiar characteristics of coned-disc springs depend solely on the various shape ratios, it is possible to use any material that exhibits constant physical properties, and there is virtually no restriction on size. These springs range from tiny stainless-steel 'snap' washers made from 0.010 mm thick material and used to monitor the effect of drugs on the 'heaviness' of a rat's footfall in a medical research laboratory, through 250 mm diameter 'constant-load' discs that enable the roof of the Melbourne Olympic Swimming Pool to swing gently in the wind controlled only by steel cables and thus afford an unobstructed view from all parts of the arena, to the gigantic 400 mm plus 'Hooke's-law' disc used as the weighing member in the well-known Craneweigher range of lifting and weighing balances, which enable the cargo of a ship to be lifted, weighed to Department of Weights and Measures standards, and loaded on to lorries for further transport, all in one movement. At the moment the heaviest of such mechanisms is capable of handling loads of 100 tonnes.

Owing to the simplification necessary to reduce the design of disc springs to a handleable level, the finished product can sometimes vary considerably from the original specification, and errors of up to 15 per cent occasionally occur. Once the design has been established however, repeatability is first class and variations should be held within the ± 5 per cent band, provided that the material thickness is maintained. Since one of the terms in the formula for load contains the third power of the material thickness, it will be apparent that general commercial tolerances on rolled or forged materials will give unacceptable load variations, and it is thus necessary to specify materials to closer limits, such as precision rolled, ground, etc.

Most manufacturers of this type of spring maintain stock lists which should be consulted before design work is attempted, since a small number of standard discs can often meet the requirements of a particular application without the need for costly development.

Fatigue

Unfortunately there is very little published data on the fatigue properties of coned discs, and the different ranges of stress to which various parts of the material are subjected make analysis highly speculative. The small amount of work done with strain gauges suggests that considerable modification to the stress pattern takes place on the first compression. The high stress localized at the upper inner edge appears to be dissipated into and supported by the surrounding materials (hence the common use of 1450 N/mm² as a static stress for steel discs although the physical properties of the material would suggest 1000 N/mm² as a more likely maximum). Undoubtedly modified Goodman diagrams could be produced for the various zones of the disc, but until they are the designer should rely on the experience of accepted manufacturers.

Diaphragm springs

A diaphragm spring is an initially coned disc in which radial slots are produced in order to provide greater flexibility. Fig. 64 shows a typical disc in which t is material thickness, D_o is outside diameter, D_r is diameter at root of slots, D_i is inside diameter, b_0 is width of 'tongue' at root, b_l is width of 'tongue' at tip, n is number of 'tongues', h_l is coned height under tips, and h_r is coned height under roots.

Diaphragm springs are used where there is ample space to accommodate a large outside diameter and engineering requirements dictate that a single spring is preferable to a pair or a stack. A typical example, and probably the most important one, occurs in the modern automobile clutch spring, where the facility for producing a load–deflection diagram with a considerable 'flat' portion enables the manufacturer to provide the driver with a light non-progressive pedal load.

Providing that the condition that the non-slotted portion of the spring acts as a normal disc is accepted (i.e. its cross-section rotates without distortion), it is a fairly simple matter to calculate the total deflection as the sum of three components: (1) that due to the disc itself, (2) that produced at the end of the fingers by the movement of the disc, and (3) the deflection of the fingers themselves acting as cantilevers.

Although it is possible to produce a general solution to the load–deflection equation, this entails solving a cubic equation, and it is much simpler to calculate the deflection in three parts as suggested in the last paragraph.

Fig. 65 illustrates the mode of deflection of the spring in which δ is the total deflection, δ_1 is the deflection of the disc, and δ_2 is the deflection caused by the cantilevers. It is obvious that δ is *not* the sum of δ_1 and δ_2.

Remembering that the load is applied at the end of the tongues and not at their roots, then

$$\delta = \delta_2 + \delta_1[\{1 - (D_i/D_o)\} \div \{1 - (D_r/D_o)\}]. \quad (40)$$

The deflection δ_2 of n cantilevers arranged in parallel in this manner is given by

$$\delta_2 = MP(D_r - D_i)^3 (1 - \mu^2)/2Et^3nb_0, \quad (41)$$

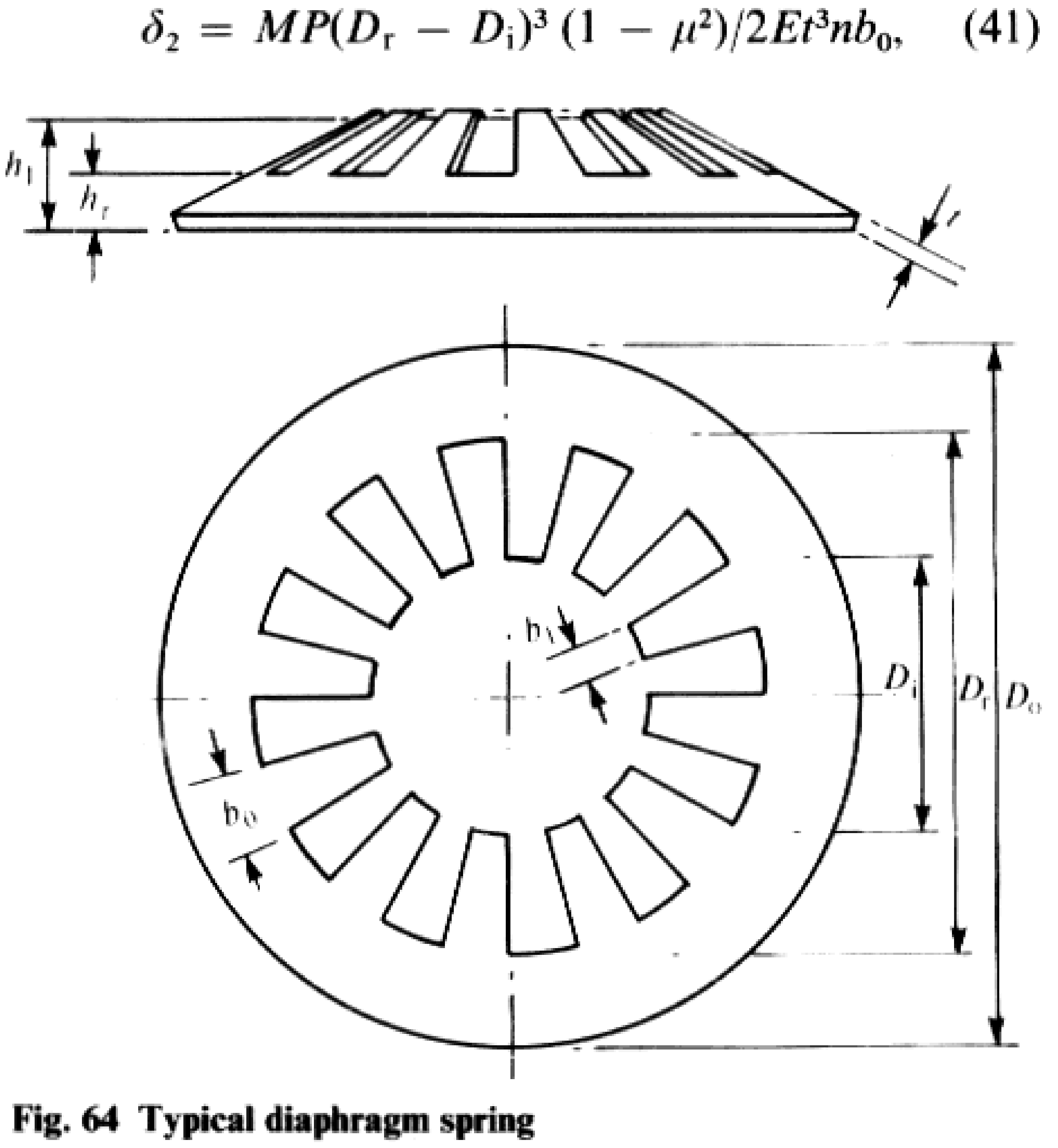

Fig. 64 Typical diaphragm spring

where E is the modulus of elasticity, μ is Poisson's ratio, and M and P are defined in the next paragraph.

M is a constant depending on the ratio b_l/b_0 according to the relationship shown in Fig. 23. The theoretical equation connecting P and δ_1 is as follows:

$$P = K_3 \times \frac{Et^3\delta_1}{(1 - \mu^2)D_o^2}\left\{1 + \left(\frac{h_r}{t} - \frac{\delta_1}{t}\right)\left(\frac{h_r}{t} - \frac{\delta_1}{2t}\right)\right\} \times \left\{\left(1 - \frac{D_r}{D_o}\right)\Big/\left(1 - \frac{D_i}{D_o}\right)\right\}, \quad (42)$$

$$\text{where } K_3 = \frac{2\pi}{3}\left(\frac{D_o}{D_r}\right)^2 \log_e\left\{\left(\frac{D_o}{D_r}\right)\Big/\left(\frac{D_o}{D_r} - 1\right)^2\right\}. \quad (43)$$

Since this is a cubic in δ_1, there is no simple means of finding a solution, although, if it is essential to do so, the method suggested on p. 28 may be employed.

Stress

Calculation of the design stresses in a simple coned-disc spring of approximately the design favoured by clutch designers shows that the maximum stress occurs at the upper edge and is approximately twice that at the lower edge. However, prolonged fatigue testing of this type of spring has shown that failure is almost invariably initiated at the lower edge. The main reason for this is the difference in the ability of carbon and low-alloy steels to withstand tensile and compressive stresses. Provided that certain design limitations are imposed (see p. 33) the designer can be sure that the maximum tensile stress will occur at the lower outer edge and that the stresses in other parts, although numerically higher, will be less harmful and may be ignored. It is permissible therefore to treat the spring as a coned disc without tongues and calculate the maximum tensile stress in the normal manner.

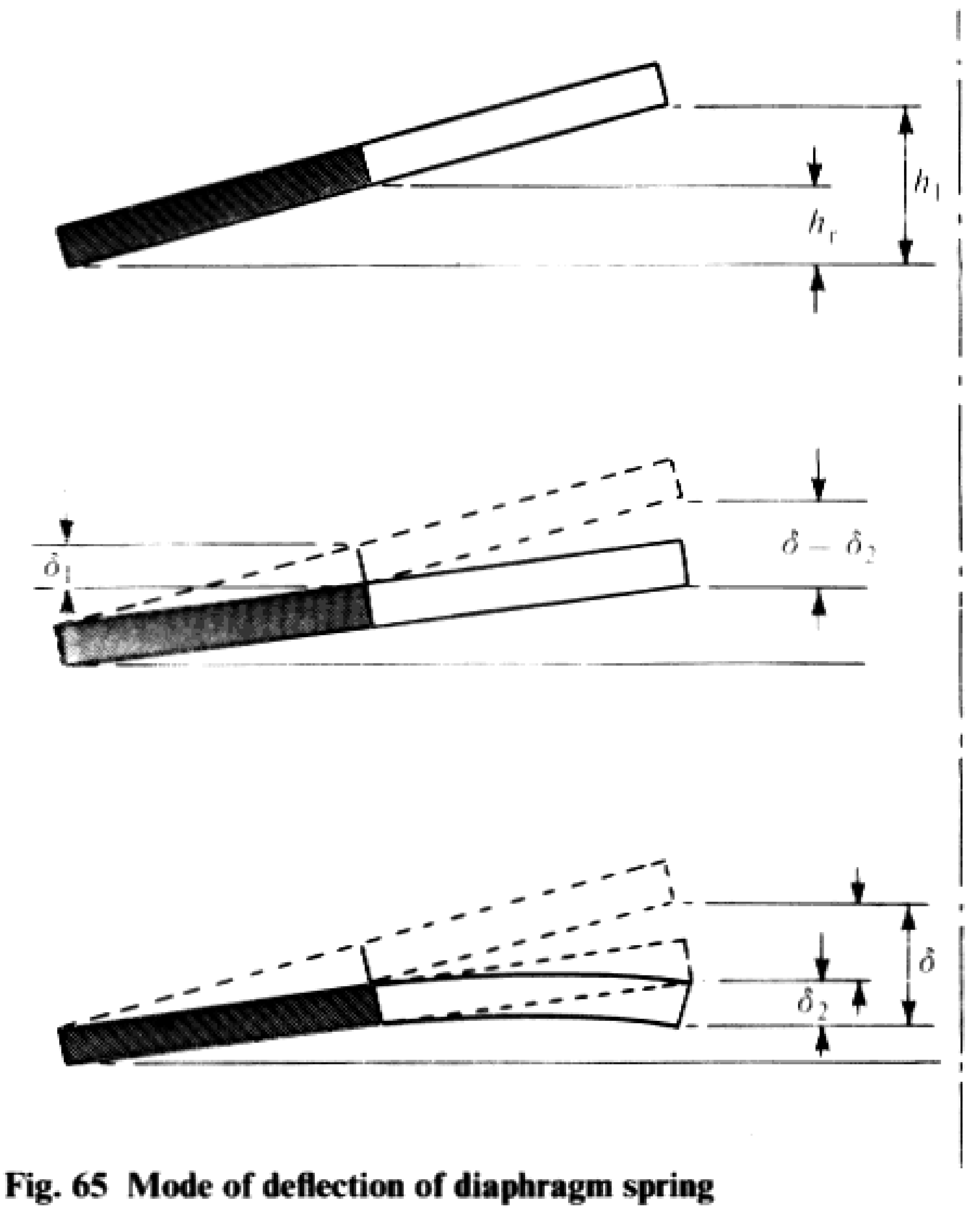

Fig. 65 Mode of deflection of diaphragm spring

In order that this method may be used it is essential that the ratio $h_r/t > 0.8$ and that the ratio $D_o/D_r \leqslant 2$. Fortunately, owing to the shape of the load–deflection characteristic required by the clutch designer, these requirements are usually met.

Design formulae

In addition to eqns (41–43) the designer will also require the following equations for stress.

$$f = \frac{Et}{(1 - \mu^2)D_o^2} \times \frac{D_r}{D_o} \times K_4\delta_1 \left\{1 + K_5\left(\frac{h_r}{t} - \frac{1}{2t}\right)\right\} \quad (44)$$

where

$$K_4 = 2\left(\frac{D_o}{D_r}\right)^2 \div \left(\frac{D_o}{D_r} - 1\right), \quad (45)$$

and

$$K_5 = 2\left\{1 - \frac{1}{\log_e (D_o/D_r)} + \frac{1}{(D_o/D_r) - 1}\right\}. \quad (46)$$

For ease of computation K_3, K_4, and K_5 are plotted as functions of D_o/D_r in Figs 66 and 67.

It appears that little fundamental research has been done into the fatigue properties of materials stressed in this manner and the fatigue diagram shown in Fig. 68 is limited to BS 1449 spring steel to CS 70 in the size range 2–3 mm. There is evidence that shot peening of the tensile face of the spring has a beneficial effect but it also presents difficulties. The effect of peening one face only disturbs the balance of stresses and, since the free height changes during the process, it is necessary to carry out one or more preliminary tests on sample springs to establish the correct pre-peened height that will give the required height after peening.

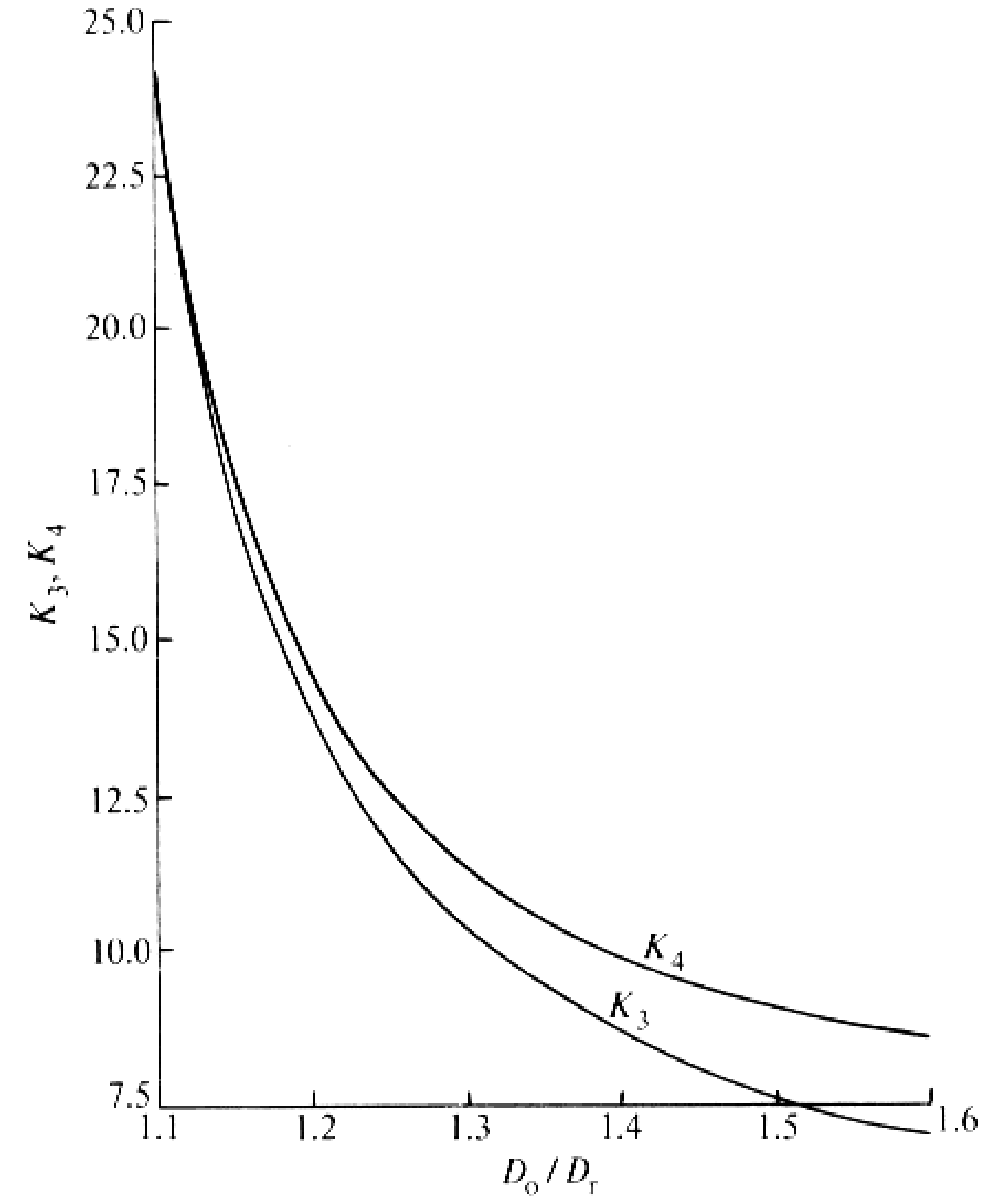

Fig. 66 Design factors K_3 and K_4 related to spring shape ratios

Worked example

It is required to design an automobile clutch spring with a 'flat' characteristic over a movement of 2.5 mm at a nominal load of 1200 N, the force not to vary by more than 200 N over the travel. An outside diameter of 150 mm can be accommodated and the minimum operating life should be 10^5 cycles. Bearing in mind the limiting values of shape ratios (i.e. D_o/D_i, D_o/D_r, h_r/t) that will produce a satisfactory spring, the procedure is to establish the material thickness in stages from the given outside diameter, and this is accomplished by means of the following numbered steps.

1. Assuming an inside diameter slightly more than half the outside diameter (say 80 mm) and allowing 'tongues' which are slightly longer than the radial dimension of the solid disc, the following diameters are obtained:

D_o = 150 mm
D_i = 80 mm
D_r = 120 mm.

These are 'round-figure' starting points to be adjusted either way if the final result is not close enough to the

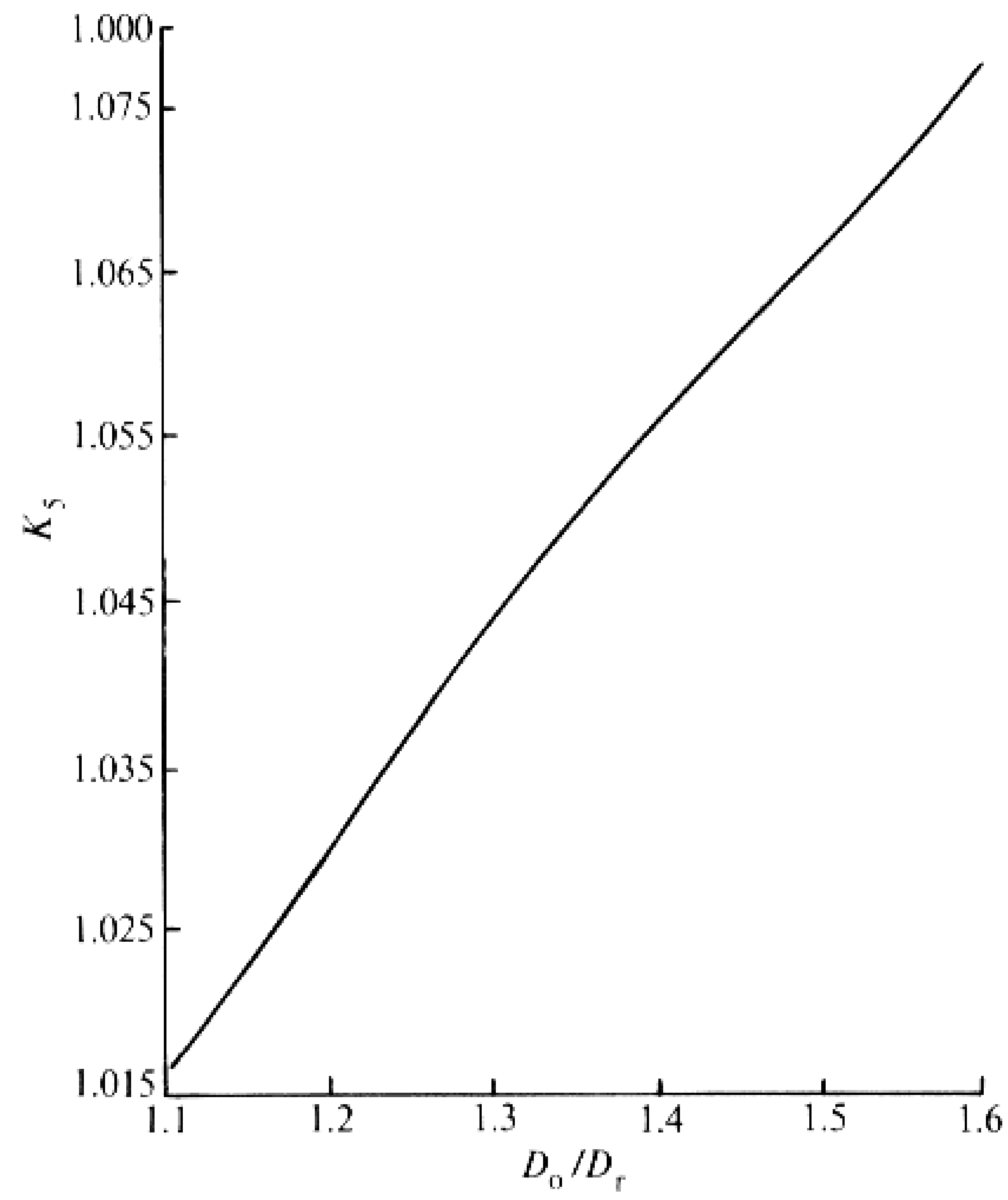

Fig. 67 Design factor K_5 related to spring shape ratios

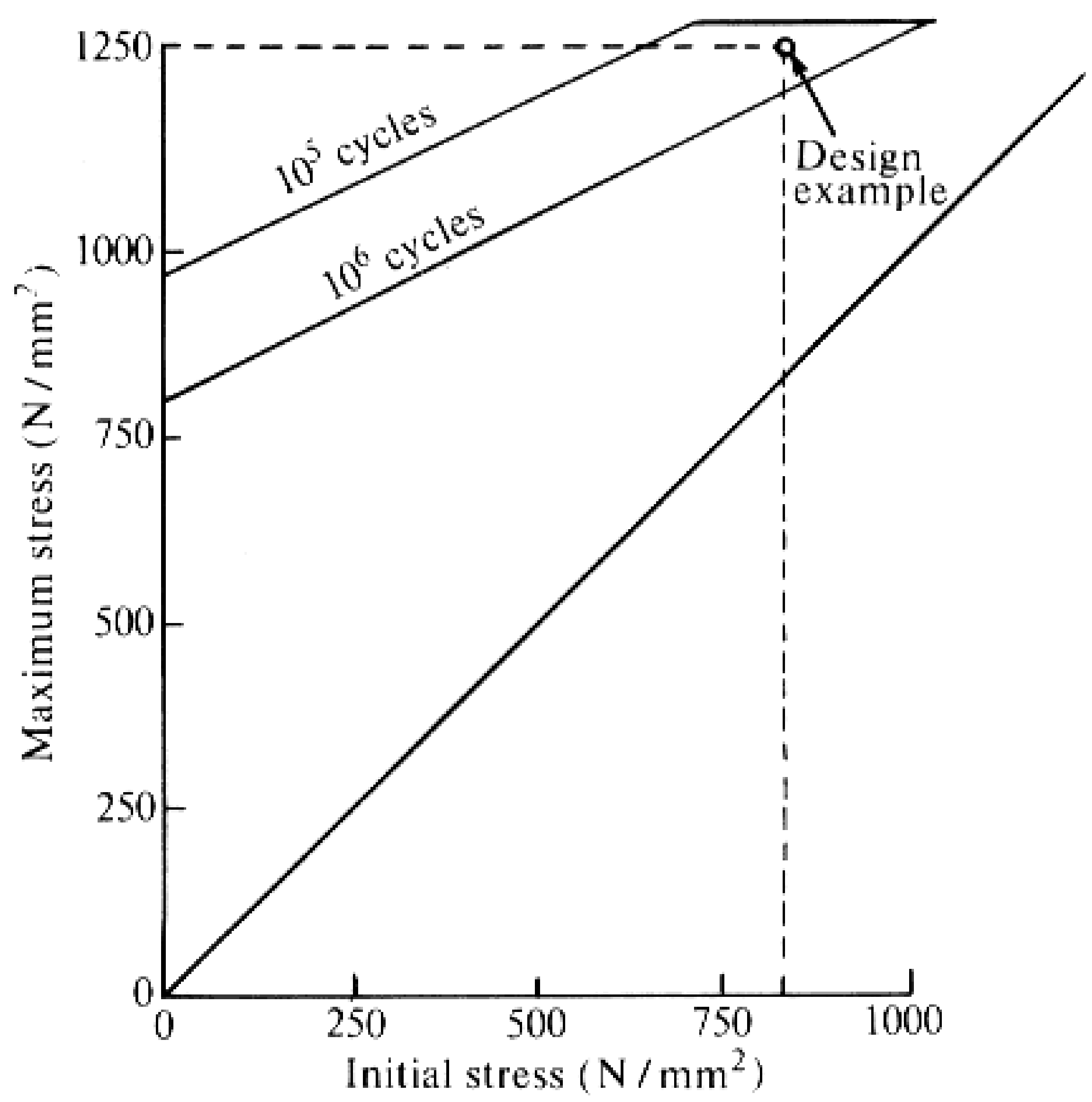

Fig. 68 Fatigue diagram for diaphragm spring

specification. Although the number of 'tongues' is not vitally important, the greater this number the more uniform the stress distribution in the transition zone from the solid disc to the tongues, where a blending radius of $1\frac{1}{2}$–$2t$ is commonly used. For the present design, 12 tongues will be selected, 8, 12, and 16 being frequently-used values.

By dividing the inside circumference to accommodate twelve leaves, each slightly larger than the complementary gap, a tip width of 9 mm is arrived at, which again is a round-figure starting point that may be adjusted either way according to the final result.

2. Since the spring is to produce a 'flat' zone of 2.5 mm, this figure represents approximately 40 per cent of the total travel, which is thus approximately 6.25 mm—the free height under tips, h_l. From the geometry of the disc $h_r = 2.7$ mm.

3. The required load–deflection characteristic dictates a height/thickness ratio of approximately 1.5 so that

$$h_r/t = 1.5$$

and

$$t = 2.7/1.5 = 1.8 \text{ mm}.$$

Once again 2 mm is used as a preferred size so that h_r becomes 3 mm.

4. From Figs 66 and 67

$$K_3 = 11.7$$

$$K_4 = 12.5$$

$$K_5 = 1.037.$$

5. Thus, from eqn (44) solid stress (with disc portion of spring deflected flat) is given by,

$$f = \{208\,000/(0.91 \times 150^2)\}2 \times 12.5 \times 3\{1 + 1.037(1.5 - 0.75)\}$$
$$= 1357 \text{ N/mm}^2,$$

which is high but not excessive, successful designs having been produced at just under 1500 N/mm².

The load–deflection curve may now be examined at several points to check that the figures obtained so far are within the specification or to decide what modification is necessary.

6. With a free height of 3 mm at the root of the tongues the standing free height from the geometry of the spring is 7 mm, giving the following provisional design:

$$D_o = 150 \text{ mm}$$
$$D_r = 120 \text{ mm}$$
$$D_i = 80 \text{ mm}$$
$$t = 2 \text{ mm}$$
$$h_l = 7 \text{ mm}$$
$$h_r = 3 \text{ mm}$$
$$n = 12$$
$$b_l = 9 \text{ mm}.$$

Using this data the loads necessary to deflect the solid disc through successive deflections (including 'flat') may now be calculated from eqn (42) to give the following results:

δ_1 (mm)	*load* (N)
0.75	755
1.0	911
1.5	1117
2.0	1215
2.5	1225
3.0	1215.

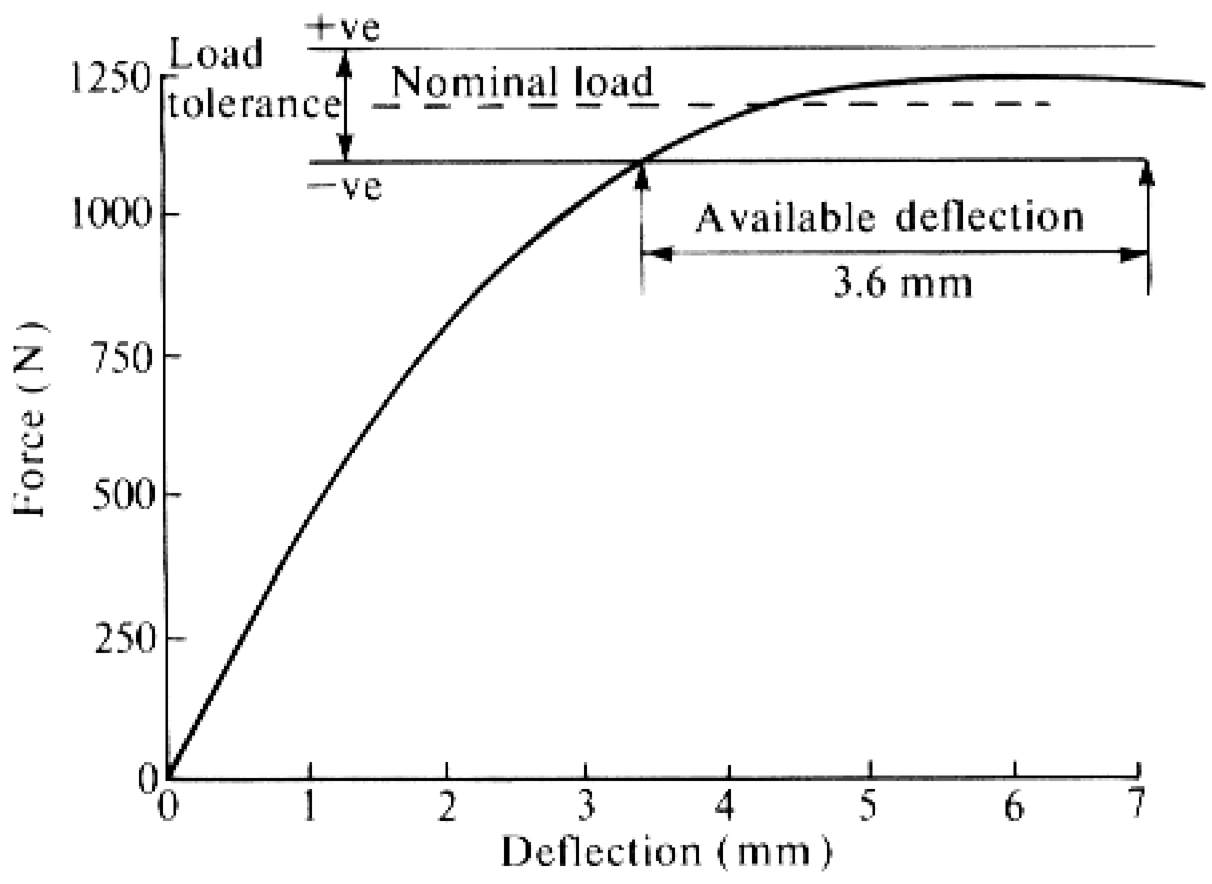

Fig. 69 Load–deflection curve from design example

7. The deflections produced in the tongues by these loads may now be calculated from eqn (41):

load (N)	δ_2 (mm)
775	0.135
911	0.163
1117	0.200
1215	0.217
1225	0.219.

8. From eqn (40):

$$\delta = 2.33\delta_1 + \delta_2,$$

giving the following co-ordinates for the load–deflection characteristic:

P (N)	δ (mm)
775	1.87
911	2.49
1117	3.70
1215	4.89
1225	6.05
1215	7.22.

These figures are plotted in Fig. 69 and show that the load tolerance can be met over a travel in excess of 3.5 mm so that the postulated design is satisfactory.

9. *Fatigue performance*. As a total deflection in operation of only 2.5 mm is needed, this may be chosen anywhere within the 3.5 mm 'flat' zone, and for this example 3.75–6.25 mm is selected.

From eqn (44) the initial and maximum stresses are 860 and 1254 N/mm², respectively, and plotting this point on Fig. 68 indicates that a life in excess of 100 000 cycles may be expected.

Ring springs

A ring spring consists essentially of a series of inner and outer rings having conically machined mating surfaces as shown in section in Fig. 70, where

D'_o is the outside diameter of the inner ring,
D'_i is the inside diameter of the inner ring,
D'_m is the mean diameter of the inner ring,
D''_o is the outside diameter of the outer ring,
D''_i is the inside diameter of the outer ring,
D''_m is the mean diameter of the outer ring,
n is the number of elements, where a single element consists of the mutually mating halves of one outer and one inner ring, giving six elements for the spring illustrated,

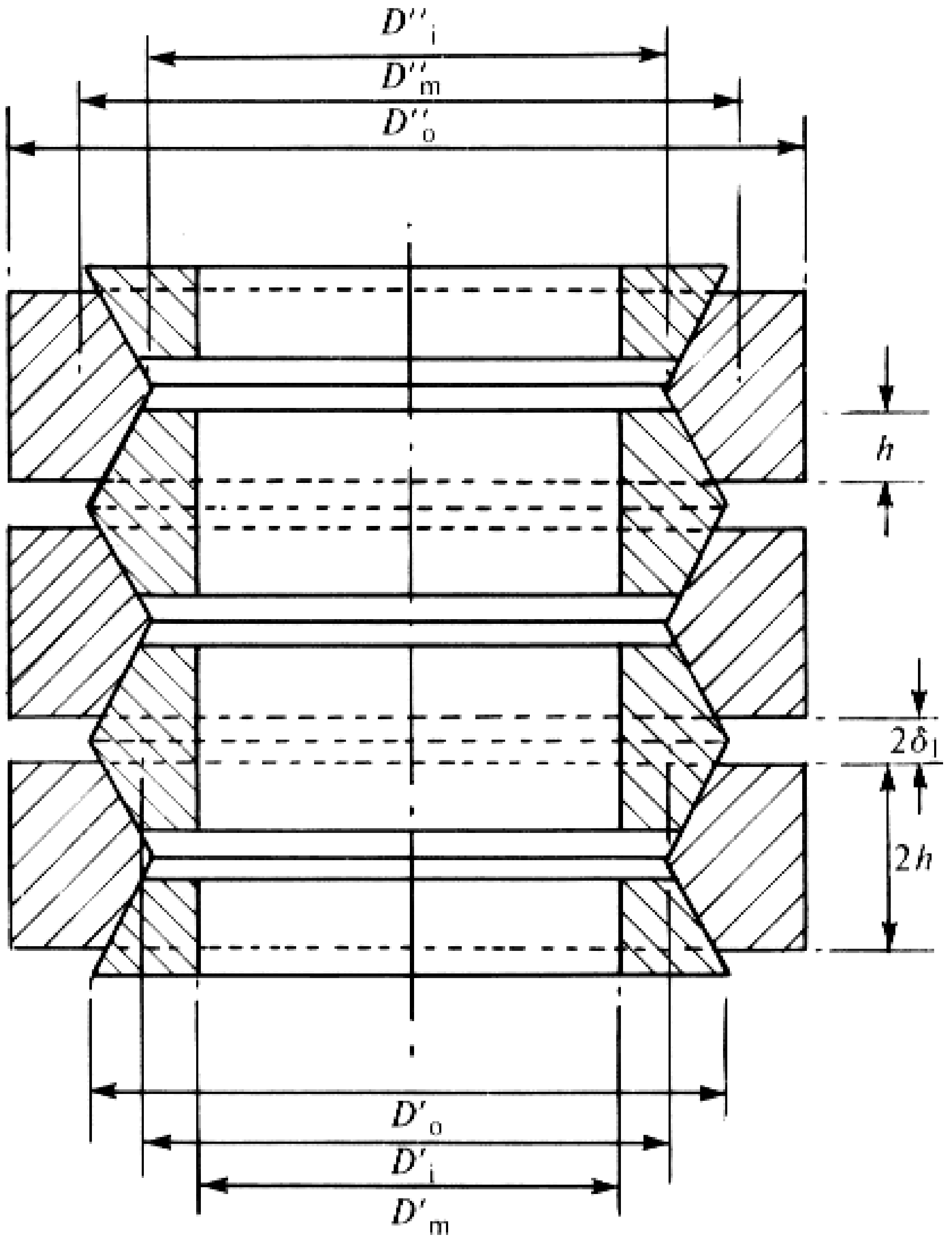

Fig. 70 Typical ring spring in section

h is the solid height of a single element and is equal to half the axial depth of one outer ring,

δ_1 is the movement from free to solid of a single element and is equal to half the gap between the outer rings.

When an axial load is applied to the spring, the inner rings are forced to contract and the outer rings to expand owing to sliding which takes place between the mating faces. Thus each element may be considered subject to a force acting normally to the mating faces and a frictional force opposing the sliding action, i.e. a normal force F and a frictional force μF where μ is the coefficient of friction between the faces. Since the frictional force always opposes motion, the load–deflection diagram of a ring spring exhibits a very large hysteresis loop as shown in Fig. 71.

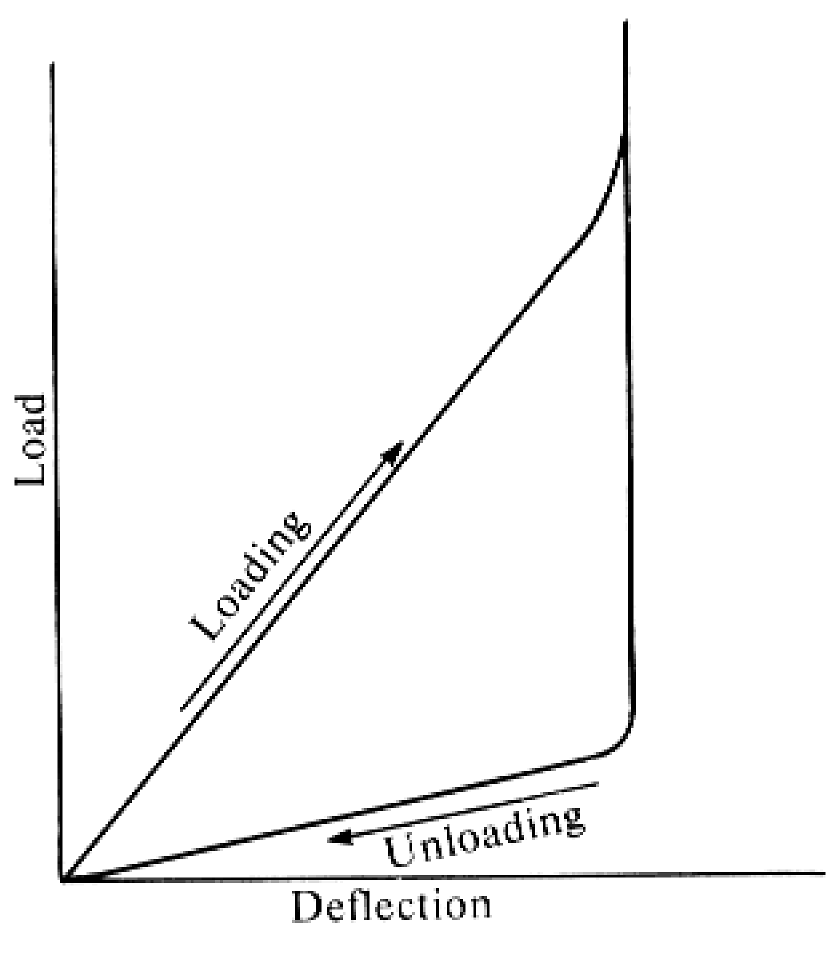

Fig. 71 Full cycle load/deflection characteristic of ring spring assembly

Design calculations

1. Stress. Consider firstly the forces acting on a single element of an *inner* ring when an *increasing* load is applied (Fig. 72).

If P'_r is the radial load per mm of the circumference described by the mean diameter D'_m when rotating about its centre, the compressive stress will be $f_c = P'_r D'_m/2A'$, where A' is the cross-sectional area of the inner ring.

By resolving the forces F and μF,

$$P'_r = 2(F\cos\alpha - \mu F\sin\alpha)/\pi D'_m,$$

and

$$f_c = 2\{(F\cos\alpha - \mu F\sin\alpha)/\pi D'_m\} \times (D'_m/2A')$$

$$= F(\cos\alpha - \mu\sin\alpha)/\pi A'. \qquad (47)$$

The axial load P_a acting on the spring is obtained by resolving the forces F and μF along the spring axis, whence

$$P_a = F\sin\alpha + \mu F\cos\alpha$$

$$= F(\sin\alpha + \mu\cos\alpha),$$

or

$$F = P_a/(\sin\alpha + \mu\cos\alpha). \qquad (48)$$

Substituting eqn (48) in eqn (47),

$$f_c = \frac{P_a}{\pi A'} \times \frac{(\cos\alpha - \mu\sin\alpha)}{(\sin\alpha + \mu\cos\alpha)}$$

$$= \frac{P_a}{\pi A'} \times \frac{(1 - \mu\tan\alpha)}{(\tan\alpha + \mu)},$$

or

$$\frac{P_a}{\pi A'} \times K_c, \qquad (49)$$

where

$$K_c = \frac{1 - \mu\tan\alpha}{\tan\alpha + \mu}.$$

The values of K_c for α between 10° and 45° and μ between 0.1 and 0.3 are shown for convenience in Table 3.

It may similarly be shown that the circumferential tension stress in the *outer* ring is

$$f_t = (P_a/\pi A'') \times K_c \qquad (50)$$

where A'' is the cross-sectional area of the outer ring.

However, when a compressive and a tensile stress act at right angles to one another, it is necessary to

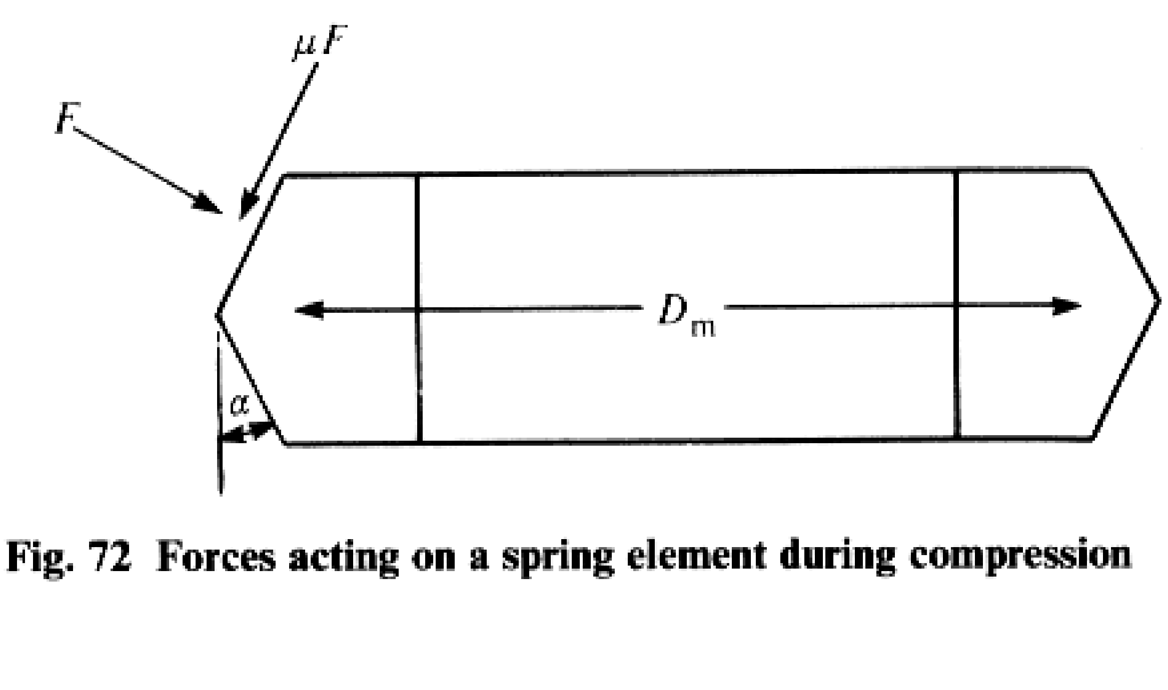

Fig. 72 Forces acting on a spring element during compression

Table 3. Values of K_c (upper rows) and K_R from α and μ.

μ \ α	10°	15°	20°	25°	30°	35°	40°	45°	
0.10	3.555	2.645	2.077	1.684	1.391	1.162	0.984	0.818	(K_c)
0.12	13.337	6.115	3.926	2.857	2.216	1.783	1.467	1.222	(K_R)
0.14	3.304	2.495	1.976	1.610	1.335	1.117	0.938	0.786	
	18.138	6.979	4.277	3.049	2.338	1.868	1.541	1.278	
0.16	3.084	2.360	1.883	1.542	1.281	1.074	0.901	0.741	
	28.228	8.110	4.692	3.293	2.472	1.960	1.598	1.326	
0.22	2.890	2.237	1.793	1.478	1.231	1.032	0.867	0.724	
	63.080	9.665	5.187	3.508	2.618	2.058	1.670	1.381	
0.18		2.125	1.718	1.417	1.183	0.993	0.833	0.695	
		11.925	5.791	3.786	2.779	2.165	1.746	1.439	
0.20		2.023	1.644	1.361	1.138	0.955	0.801	0.667	
		15.517	6.541	4.105	2.956	2.279	1.827	1.500	
0.22		1.929	1.575	1.308	1.095	0.919	0.770	0.639	
		22.107	7.501	4.501	3.154	2.402	1.913	1.576	
0.24		1.842	1.511	1.257	1.054	0.885	0.740	0.613	
		38.147	8.769	4.913	3.375	2.537	2.005	1.632	
0.26			1.451	1.219	1.015	0.852	0.711	0.587	
			10.525	5.435	3.625	2.685	2.104	1.703	
0.28			1.395	1.165	0.978	0.820	0.684	0.563	
			13.118	6.069	3.907	2.846	2.208	1.778	
0.30				1.122	0.942	0.790	0.657	0.538	
				6.854	4.231	3.024	2.322	1.857	

compound them into a single *equivalent* stress and this is achieved by arithmetically adding their individual values.

For the outer ring, there is a compressive stress due to the normal force, F, which must be added to the tensile stress f_t calculated from eqn (50), whereas for the inner ring both stresses are compressive and no addition is necessary. The additional stress for the outer ring may be calculated as follows.

The radial load per mm of circumference of the outer ring, P''_r, may be calculated in the same way as for the inner ring by substituting D''_m for D'_m, thus

$$P''_r = 2F(\cos\alpha - \mu\sin\alpha)/\pi D''_m,$$

or

$$F = P''_r \pi D''_m/2(\cos\alpha - \mu\sin\alpha), \qquad (51)$$

and the circumferential *tension* stress by,

$$f_t = P''_r D''_m/2A''. \qquad (52)$$

Thus

$$F = (2A''f_t/D''_m) \times \pi D''_m/2(\cos\alpha - \mu\sin\alpha)$$

$$= \pi A'' f_t/(\cos\alpha - \mu\sin\alpha). \qquad (53)$$

Now if the projected length, on the spring axis, of the contact area between the inner and the outer springs at the axial load, P_a, and stress, f_t, is b, then the average *compression* stress in the contact area is given by

$$f_c = F\cos\alpha/\pi Db \text{ where } D = D''_o + D'_i.$$

Expressing this in terms of f_t,

$$f_c = f_t \times A''/Db(1 - \mu\tan\alpha). \qquad (54)$$

2. *Deflection.* In order to calculate the axial deflections of the spring it is first necessary to calculate the radial deflections of both inner and outer rings, δ'_r and δ''_r. Since *stress/strain* equals the *modulus of elasticity*, E, the deflection of the inner ring

$$\delta'_r = Df_c/E.$$

As each ring has two conical faces the axial deflections caused by the contraction of the inner and outer rings, respectively, are given by

$$\delta'_a = 2f_c \times n'/E \times D_i/2\tan\alpha$$

$$= f_c n' D'_i/E\tan\alpha,$$

and

$$\delta''_a = 2f_t \times n''/E \times D_o/2\tan\alpha$$

$$= f_t n' D''_o/E\tan\alpha,$$

where n' and n'' represent the numbers of elements in the inner and outer rings. Now if $n' + n'' = n$, then n is also the number of 'elements' in the spring. Total axial deflection is given therefore by

$$\delta_a = Dn(f_c + f_t)/2E\tan\alpha. \qquad (55)$$

Substituting for f_c and f_t from eqns (49) and (50),

$$\delta_a = \frac{Dn}{2E\tan\alpha}\left(\frac{P_a K_c}{\pi A'} + \frac{P_a K_c}{\pi A''}\right)$$

$$= \frac{DnP_a}{2\pi E\tan\alpha}\left(\frac{1}{A'} + \frac{1}{A''}\right) \times K_c. \qquad (56)$$

On consideration of the return stroke it may similarly be shown that

$$\delta_a = \frac{DnP_a}{2\pi E \tan \alpha}\left(\frac{1}{A'} + \frac{1}{A''}\right) \times K_R, \quad (57)$$

where

$$K_R = \frac{1 + \mu \tan \alpha}{\tan \alpha - \mu}.$$

In both cases it must be remembered that P_a has been derived from stress and therefore must have the same basic units. Thus if stress is in N/mm² then P_a is in Newtons, and the formulae require modification when kgf is used as the unit of load.

Hence if P_c and P_R are the respective axial loads at the same deflection on the compression and return strokes then

$$P_c K_c = P_R K_R,$$

or

$$P_c/P_R = K_R/K_c. \quad (58)$$

For convenience, values of K_R against α and μ are plotted in Table 3 under the corresponding values of K_c.

Practical design considerations

In order to ensure a reasonably efficient use of the spring material the following guidelines should be observed.

1. The maximum deflection of the whole spring assembly should not be greater than 20 per cent of its free height.
2. The axial height of individual rings should lie between one sixth and one fifth of the outside diameter.
3. The angle of taper should lie between 10° and 25°, although special requirements of energy absorption etc. may require the latter figure to be exceeded.
4. Radially thin rings are preferred to radially thick ones.
5. Although it is obviously possible from the formulae to calculate the exact relationship between the dimensions of the inner and outer rings, it is not an uncommon practice where the first three conditions are met to calculate the cross-sectional area of the inner ring and to add 10 per cent to it to obtain the cross-sectional area of the outer ring.
6. Where steel rings are machined all over after heat-treatment to remove decarburization and surface imperfections, stresses up to 1400 N/mm² may be employed.

Worked example

It is desired to design a ring spring having a load/unload characteristic of 11/4 to carry a load of 500 000 N. The material has a coefficient of friction, μ, of 0.14 and may be subjected to a maximum stress of 1300 N/mm². Movement under load should be approximately 40 mm and space restrictions impose a limit of 175 mm on outside diameter.

The first step is to establish the face angle α which will give the required characteristic in conjunction with the given coefficient of friction. Entering Table 3 at $\mu = 0.14$, a value of α of 15° gives $K_R/K_c = 6.98/2.50 = 2.79$, which approximates to the required value of 11/4. In view of the possible variation in μ this value of α is acceptable.

Table 4. Cross-sectional area of inner ring in mm² to carry 50 000 N at 1250 N/mm²

μ \ α	10°	15°	20°	25°	30°	35°	40°	45°
0.10	507	376	296	239	198	165	139	116
0.12	469	355	282	230	190	159	134	112
0.13	438	336	268	219	183	153	128	107
0.14	411	318	256	210	176	147	123	103
0.16	387	303	245	202	168	142	118	99
0.18	365	288	234	194	162	136	114	95
0.20	345	274	224	186	156	131	109	91
0.24	327	262	215	179	150	125	105	88
0.26	311	251	206	173	145	121	101	84
0.28	297	241	199	166	139	116	97	80
0.30	282	230	191	160	135	112	94	77

The next step is to establish the cross-sectional area of the inner ring from Table 4. A load of 500 000 N at 1250 N/mm² requires an area of 335 mm², and since this stress is below but reasonably close to the specified maximum it can be accepted as a first estimate.

Assuming the outside diameter to be approximately 170 mm, the second guideline gives an approximate axial ring height of 30 mm, and simple geometry an 'eaves' height (see Fig. 73) of 9 mm to the nearest whole millimetre. With a given angle of 15 degrees the cross-sectional area of the inner ring is then

$$(30 \times 9) + \{(15 \tan \alpha) \times 30/2\} = 330.3 \text{ mm}^2,$$

which is acceptable.

Using the 10 per cent rule, the cross-sectional area of the outer ring needs to be approximately 363 mm², giving an 'eaves' height of 10.1 mm.

Guideline 1 states that the deflection of a single ring should not exceed 20 per cent of the free height, so that the space between rings should not exceed 20 per cent of 30 mm, i.e. 6 mm.

As a first trial, 5 mm used in conjunction with an outer-ring outside diameter of 170 mm establishes an inside diameter of 125 mm and a 'mean' of 147.5 mm, giving (from eqn (56))

$$\delta_a = \frac{147.5}{2\pi E} \times 500\,000\left(\frac{1}{330.3} + \frac{1}{363}\right) \times 2.36 = 2.8 \text{ mm}.$$

Since the space available for movement is only 2.5 mm, the mean diameter must be reduced in the ratio 2.5/2.8 i.e. to 132 mm, giving a final design of $D_i = 109.5$ mm (110 mm) and $D_o = 154.4$ mm (155 mm).

To establish the number of elements required to produce a movement of 40 mm, proceed as follows:

(i) movement per element = 2.5 mm,
(ii) number of elements = 40/2.5 = 16, giving
(iii) free height = 16 (15 + 2.5) = 280 mm.

The dimensions of the spring as designed are shown in Fig. 73.

Applications and materials

The distinguishing characteristics of ring springs, namely absorption of shock loads with low recoil and a very high capacity/weight ratio, would seem to make them the obvious choice for such applications as the operating parts of heavy earth-moving equipment, but for some reason they have not achieved the popularity with British designers that

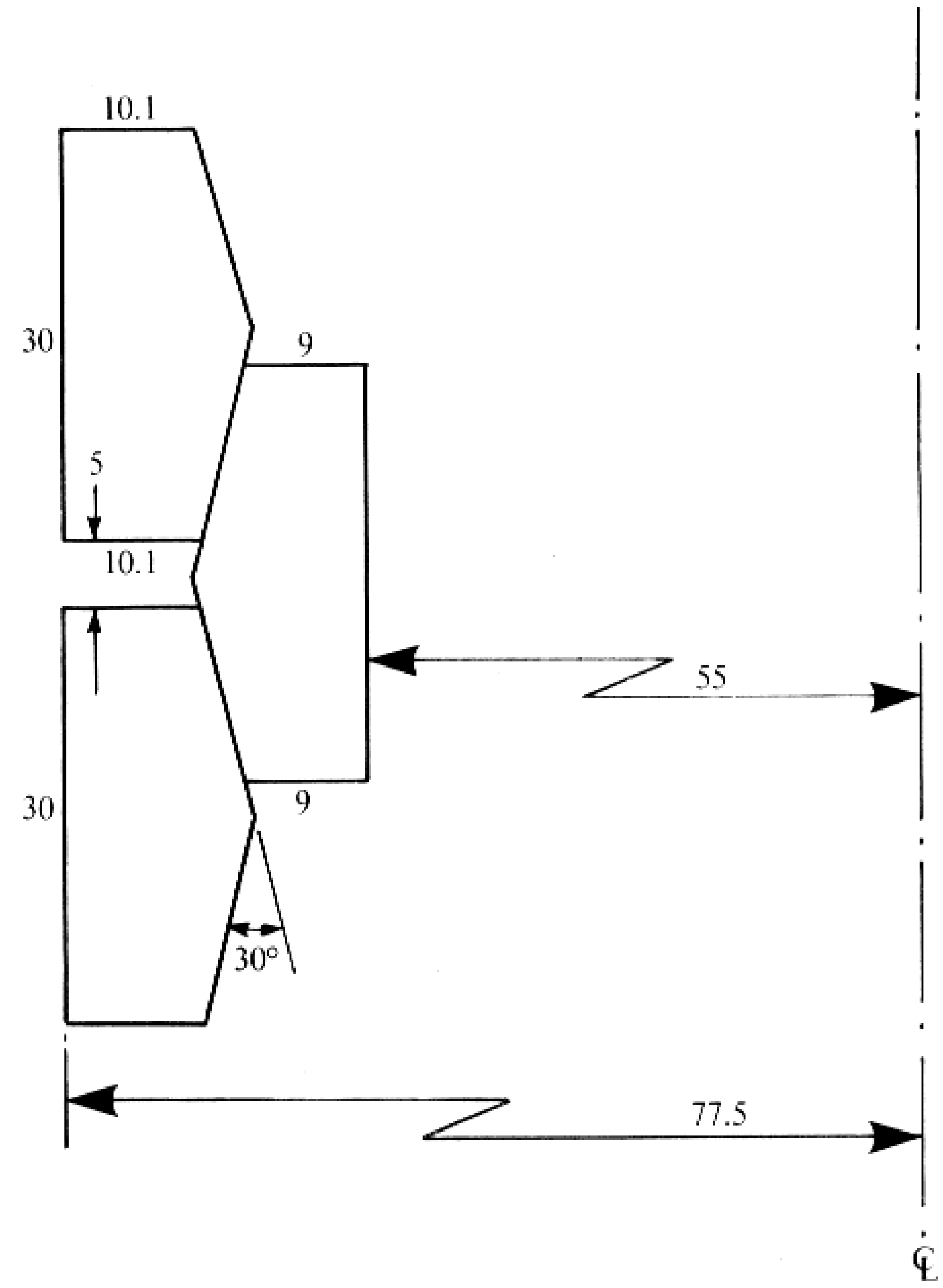

Fig. 73 Dimensions of ring spring in mm from a specific design

they deserve. In fact, at the time of writing, the author is not aware of a single British spring manufacturer who includes the design and production of ring springs as part of his normal work, the nearest approach being a single British company who import and factor continentally-produced ring springs.

Any of the standard heavy spring steels, silicon–manganese, carbon–chrome, or nickel–chrome–molybdenum are suitable for the manufacture of ring springs, while stainless, corrosion resistant, and non-magnetic alloys may all be used with suitable design.

In view of the relative simplicity of the necessary design work, it is difficult to understand the wide-spread British reluctance to use ring springs, possibly on account of the employment of friction as one of the operating characteristics. The fact remains, however, that ring springs have a widespread use on the continent, and where quantity requirements are high enough to offset the high original tool costs consideration should be given to their possible use.

Ring springs are supplied in 'off-the-shelf' packs for a wide variety of applications as shown in Fig. 74, which is a typical assembly for arresting the overrun of railway rolling stock. A suitable number of standard rings is assembled into a pair of retaining cups (A) and pretensioned to a minimum of 5 per cent of the pack capacity by means of a central bolt (**B**). The assembly which is normally referred to as a 'cartridge' is located inside the static sleeve (C) by means of an annular recess (D). The outer or dynamic sleeve is slid on to the static sleeve and pushed on far enough for a retaining ring (E) to be inserted in its groove. The buffer face is now located by means of the integral bolts (F), and sufficient force is applied to compress the cartridge while the retaining nuts are tightened.

With probably a dozen sizes of rings, each of which may be obtained with a variety of included angles, the number of combinations is only limited by the number of available housings. Where special requirements of load and stroke preclude the use of standard fittings, sleeves and shims enable the designer to meet his specification.

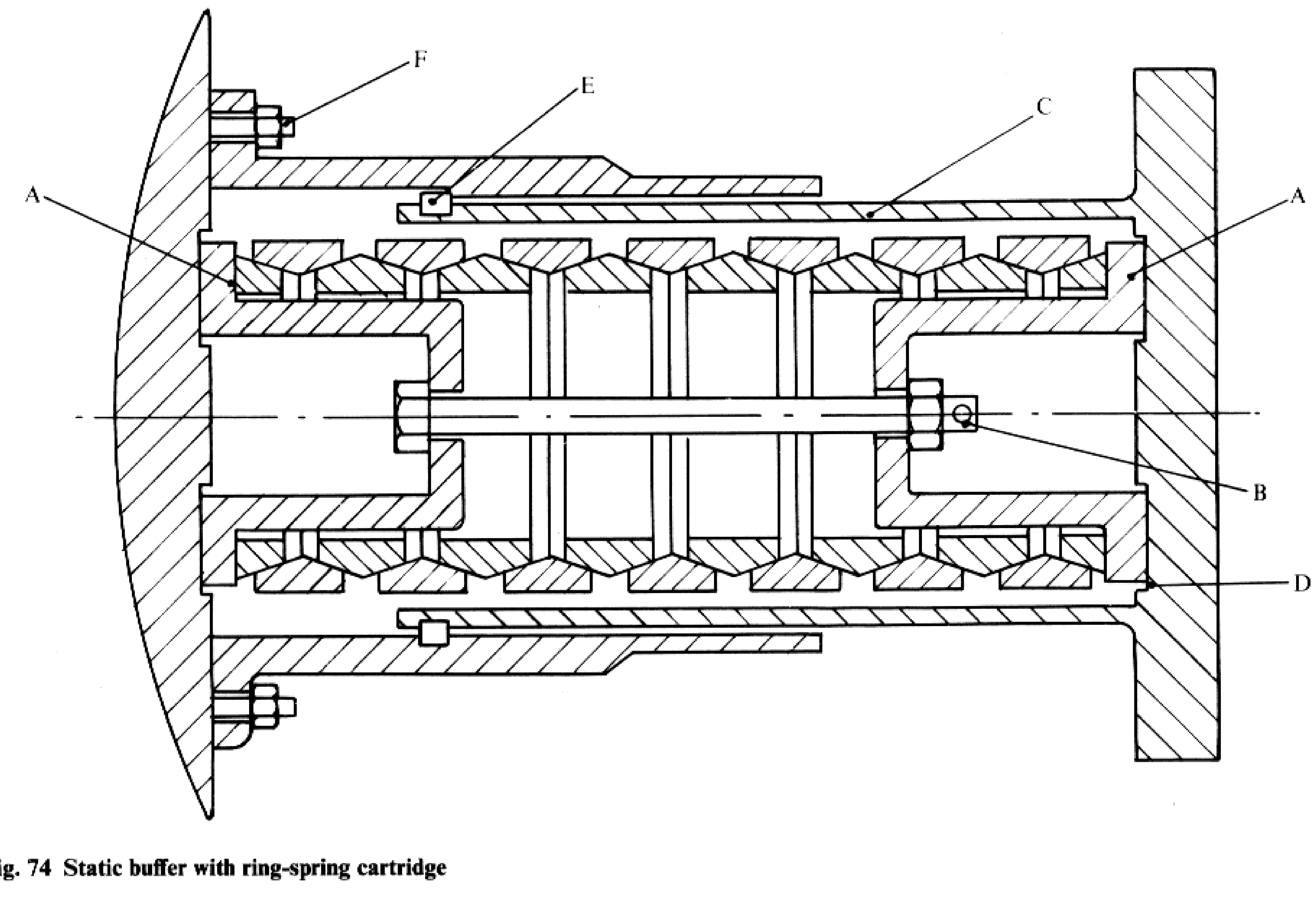

Fig. 74 Static buffer with ring-spring cartridge

Similar assemblies are in wide use in factories, rolling mills, etc., where it is necessary to arrest heavy moving equipment without a high proportion of recoil.

Lubrication. Because most ring-spring assemblies absorb approximately 60 per cent of the energy input by means of friction, it is essential that the frictional properties of the mating faces remain reasonably constant. With this end in view the continental manufacturers have developed their own range of lubricants with which the assemblies are packed during manufacture, and they claim that their lubricants never need replacing as their life exceeds the expected life of the springs.

Temperature. Ring-spring assemblies, properly lubricated, will operate consistently in temperatures up to the melting point of the lubricant provided that the rise in temperature caused by the absorption of 60 per cent of the energy input is allowed to dissipate before the next impact takes place. It will thus be seen that there is a limit to the number of full operations per minute to which a buffer may be subjected, and that this number may be increased as the proportion of total stroke employed decreases.

Fig. 75 shows the typical form of a frequency/percentage-of-stroke curve for an assembly having a permissible 20 second cycle at full stroke, from which it will be observed that the relationship is approximately logarithmic.

Fatigue life. Commercial-quality ring springs in which the elements are rolled have very different frictional characteristics from those which are fine machined and this is reflected mainly as a difference in fatigue life. In general machined assemblies may be expected to have two-and-a-half times the life expectancy of those made from rolled rings.

Caution. Ring-spring assemblies may become jammed when they are subjected to excessive shock loading, one or more pairs of rings failing to free when the load is removed. A sharp blow with a hammer is usually sufficient to free the rings, but this should never be done unless the pre-stressing bolt is in position. In common with the coned-disc spring the energy release when struck may be almost explosive, possibly resulting in severe injury or extensive damage to the equipment.

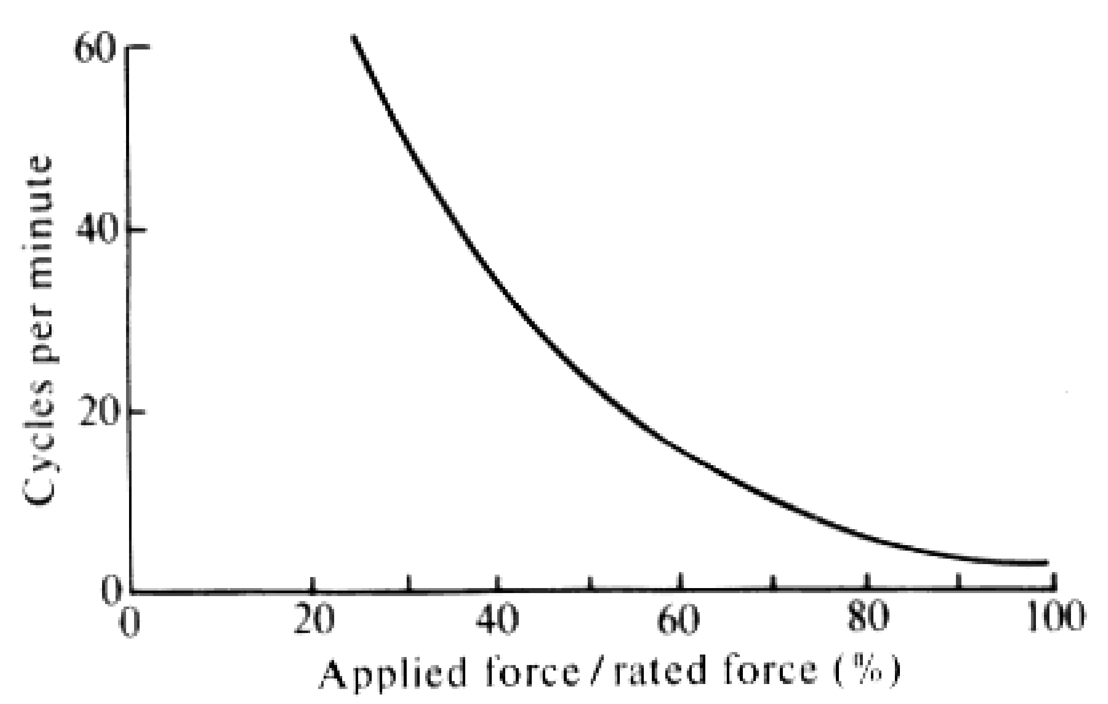

Fig. 75 Effect of operating frequency on permissible maximum force

Volute springs

Spring Research and Manufacturers' Association (1968) defines a volute spring as:

A spring produced from flat-section material, helically coiled with its thickness in the radial dimension such that each coil nests within its adjacent larger coil.

This is the generally accepted definition in the heavy spring industry, although there is no reason why the material should not be of square or even circular section. This Guide, however, is concerned almost entirely with volute springs as defined by SRAMA with materials of other sections being considered only as modifications (see Fig. 76).

The developed 'blade' used in the conventional form of a volute spring is shown in Fig. 77.

After coiling, the helix angle of the spring will either be constant or variable and the spring characteristic will depend upon the amount and nature of the variation.

The two great advantages of a spring coiled in this manner are (1) that when compressed solid its 'envelope' contains a very high proportion of material so that its volume efficiency is high, and (2) that variations in the ratio of the angles and the radii of curvature of the coils will produce a wide variety of load–deflection characteristics. A further advantage is that it is possible to induce high levels of beneficial pre-stress in the material by scragging the springs 'inside-out' into a hollow cup so that the deflection so obtained exceeds the difference between the free and solid heights. Unfortunately, it is virtually impossible to manufacture the spring so that the line of action of the load is along the axis of the spring, with the result that the stress distribution is uneven and some

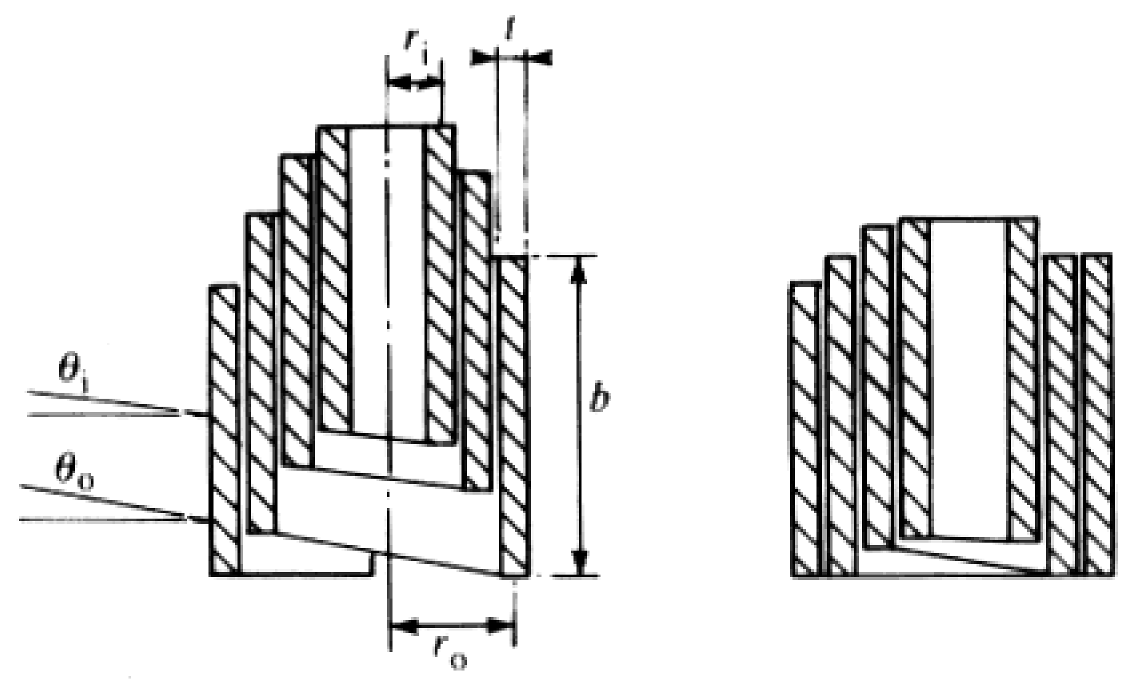

Fig. 76 Section through volute spring showing bottoming of coils during compression (for meanings of symbols see *Notation* (p. 40))

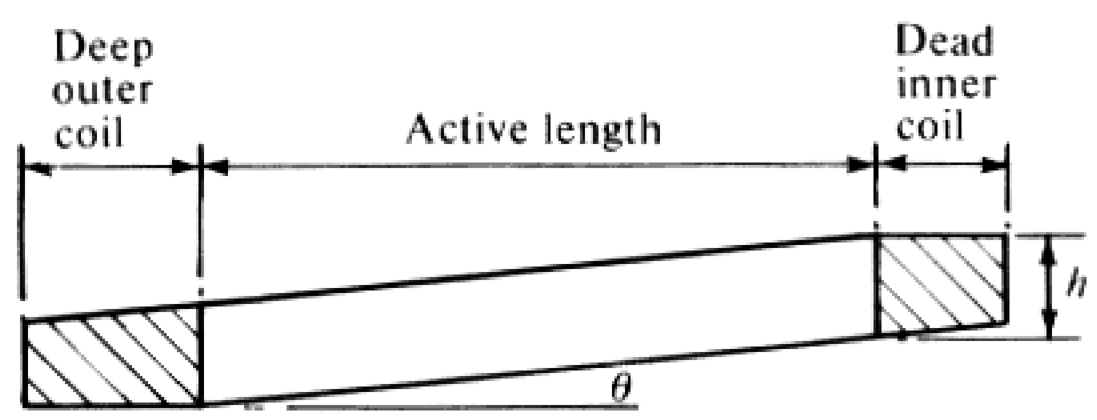

Fig. 77 Developed blade for conventional volute spring

efficiency is lost. In addition, the springs need special heavy-duty coiling equipment and a high degree of skill to ensure successful manufacture, while the large number of variables make the design mathematics exceedingly complex.

This latter difficulty is usually overcome by 'pairing' the variables and designing charts which, although causing a slight loss of accuracy, reduce the amount of design time considerably. This loss of accuracy is probably of very little importance since it is of the same order as the manufacturing tolerances that must be allowed, while the limited number of manufacturers willing to undertake such work will draw upon their own specialized experience to make whatever tooling modifications are necessary.

The desired load–deflection characteristic may be chosen from Figs 78, 79, or 80 which are non-dimensional diagrams with load plotted against deflection. The special units used are given under *Notation*.

The auxiliary factor Y which is to be used in the relationships for load and deflection may be obtained from Fig. 81.

Notation

b = height of section
t = thickness of section
r_o = radius of largest free (active) coil
r_i = radius of smallest free (active) coil
δ_t = total deflection from free to solid
W = load at a deflection of $\delta/2$ from free
σ_{max} = maximum stress
θ_o = helix angle at largest free coil radius (radians)
θ_i = helix angle at smallest free coil radius (radians)
n = number of free coils
N = total number of coils
G = torsional modulus of rigidity

Pairings

$$A = r_i/r_o \quad (59)$$
$$B = \{1 - (\theta_i/\theta_o)\} \div \{(1 - (r_i/r_o)\} \quad (60)$$
Y = deflection factor as a function of A and B
$$Z = \text{load factor} = 0.1 - 0.009A.\dagger \quad (61)$$

Relationships

$$W = YZ \times Gbt^3\theta_i/r_o^2 \quad (62)$$
$$\sigma_{max} = Gt\theta_i/Ar_o \quad (63)$$
$$\delta_t = nr_o\theta_i Y. \quad (64)$$

Simple design check ‡

(1) Data:

r_o = 95 mm n = 4
r_i = 50 mm G = 79 000 N/mm²

†Although the true plot of Z is curved, the errors introduced by the linear form given are insignificant compared with manufacturing tolerances.
‡Where for example the design is specified by the customer.

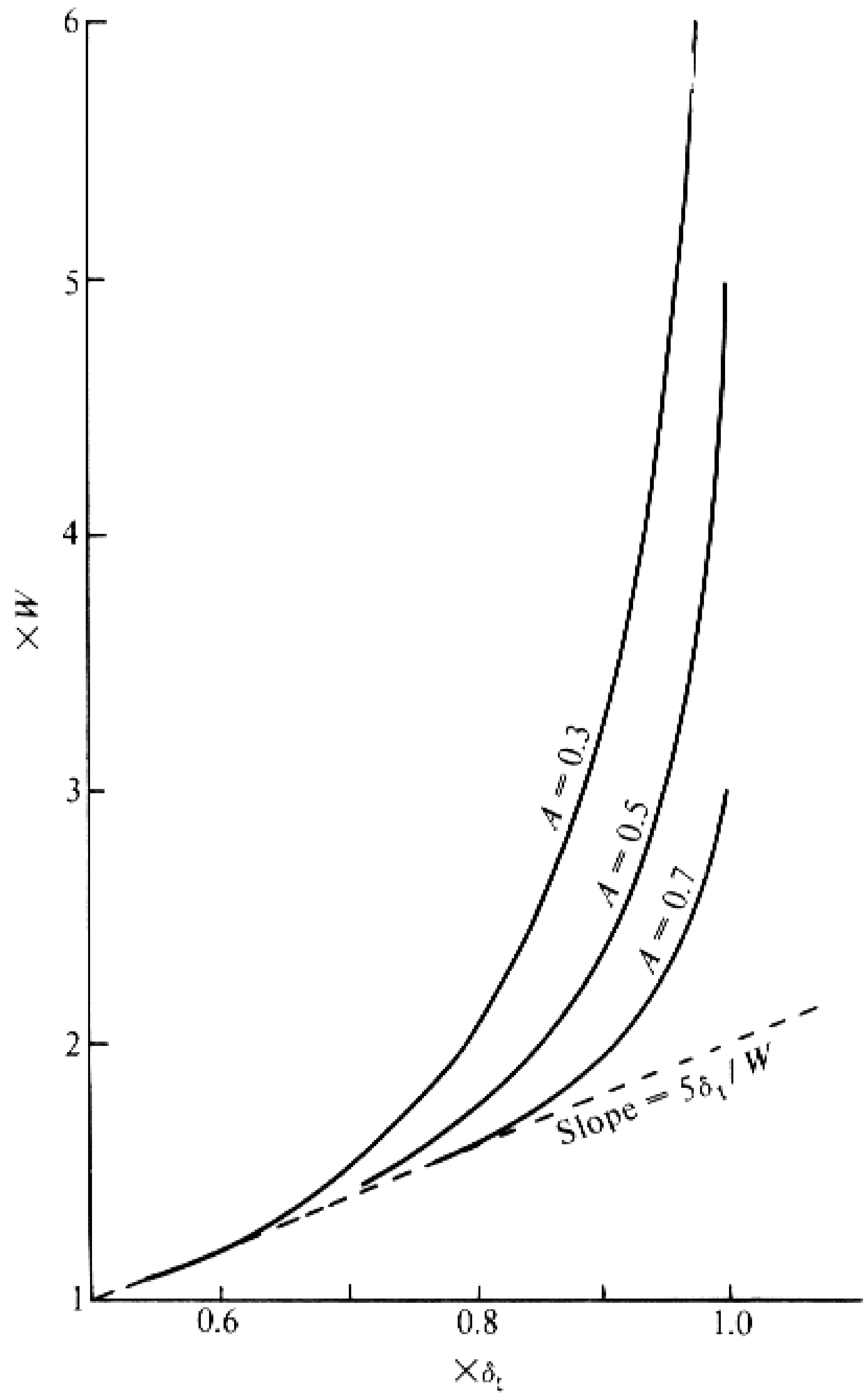

Fig. 78 Load plotted against deflection for B = 0 (for meanings of symbols see *Notation* and *Pairings*)

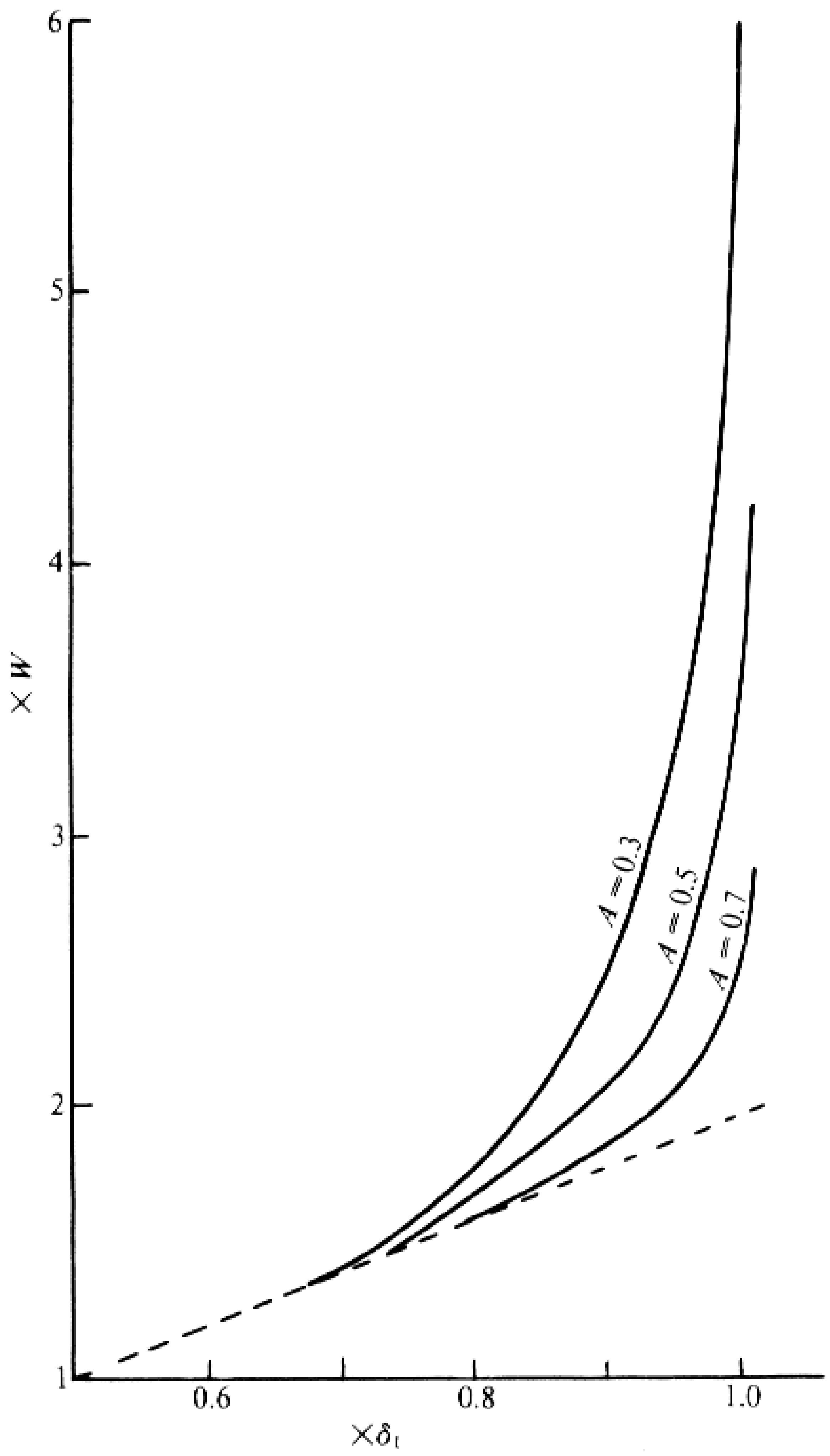

Fig. 79 Load plotted against deflection for $B = \frac{1}{2}$

$b = 190$ mm $\quad \theta_o = 0.076$

$t = 10$ mm $\quad \theta_i = 0.060$

$N = 5\frac{1}{2}$.

(2) Pairings:

$A = 0.526 \quad B = 0.445$

$Y = 5.6 \quad Z = 0.053$.

(3) Total deflection:

$$\delta_t = 4 \times 95 \times 0.06 \times 5.6 = 128 \text{ mm.}$$

(4) Maximum stress:

$$\sigma_{max} = 79\,000 \times 0.06 \times 10/(0.526 \times 95) = 949 \text{ N/mm}^2.$$

(5) Load at half deflection:

$$W = 79\,000 \times 190 \times 1000 \times 0.06 \times 5.6 \times 0.053/95^2 = 29\,618 \text{ N.}$$

(6) Load at full deflection (from Fig. 79)

$$= 29\,618 \times 3.25 = 96\,260 \text{ N.}$$

Modifications

If square-section material is used, eqn (62) must be multiplied by 0.44 and eqn (63) by 0.67.

If round-section material is used and b and t are replaced by the wire diameter, the load formula must be multiplied by 0.31 and the stress formula by 0.50.

Designing to specification

Because 'pairings' have taken place to establish the design formulae, it is necessary for the designer to make an intelligent guess at one or more of the dimensions, work out a preliminary design, and then modify it according to the dictates of movement or stress.

Usually the values of solid height, total movement, load at half movement, and maximum stress are already laid down, while the outside diameter is limited by space restrictions. Although the design may now be approached in several different ways, the following is one of the simplest procedures.

1. Although it is possible by mathematical manipulation of the three major equations to arrive at a reasonable estimate of θ_i, it will be found that this is usually of the order of 0.06 rad and it is generally simpler to use this figure as a starting point and adjust after the first trial calculation if necessary.
2. Estimate r_o, remembering that there will be 'dead' material outside this which must be accommodated in the available space.
3. Estimate r_i, remembering that the spring must be coiled on a mandrel capable of withstanding the coiling loads.
4. Calculate $A = r_i/r_o$.
5. From the two load requirements choose a load–deflection curve from Figs 78, 79, or 80, and establish B and hence θ_o.
6. Read off factor Y from Fig. 81.
7. Calculate $Z = 0.1 - 0.09A$, and
8. Substitute (i) in eqn (62) to obtain the maximum value of t that will move the distance at this stress,

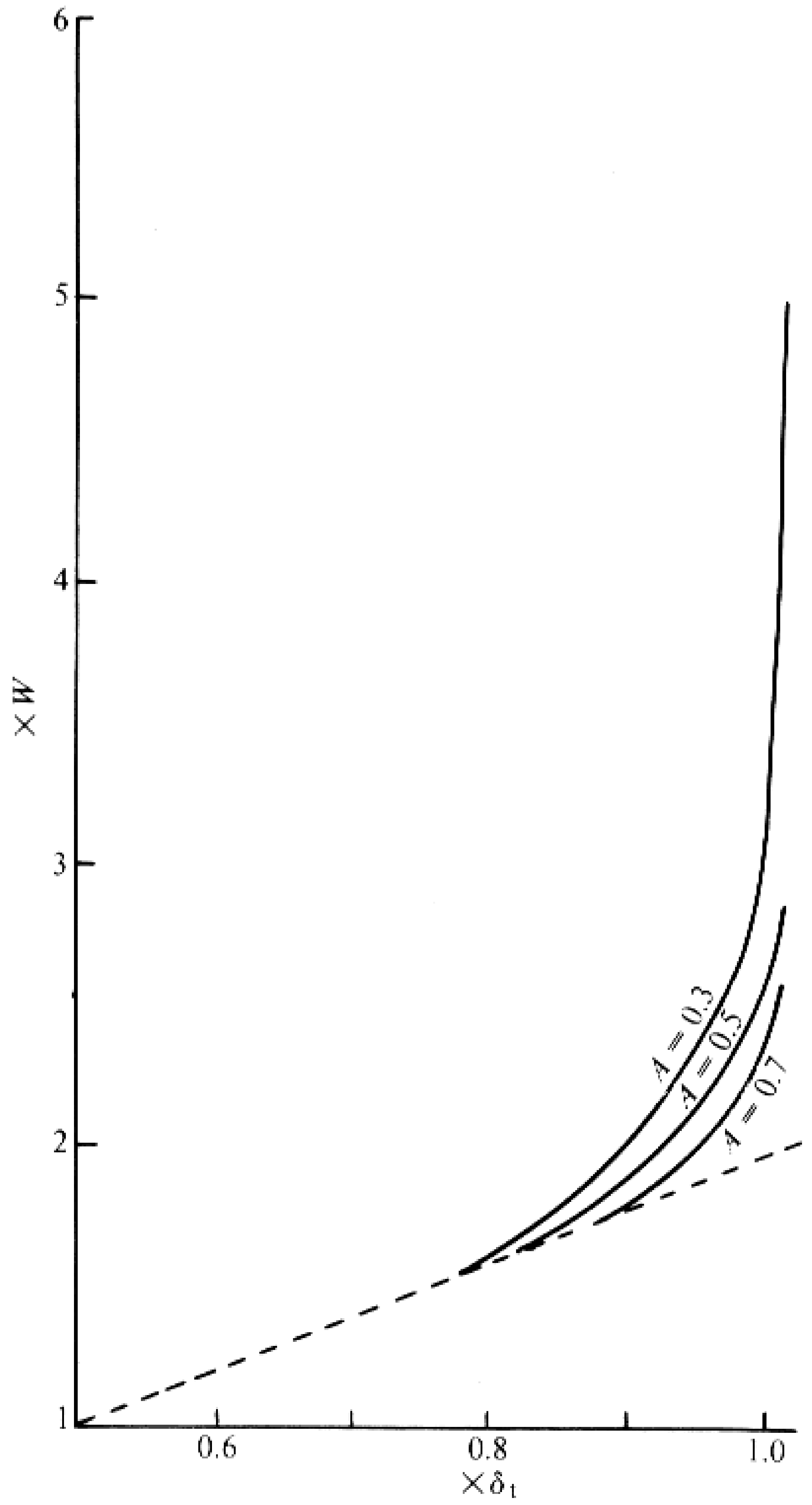

Fig. 80 Load plotted against deflection for $B = 1$

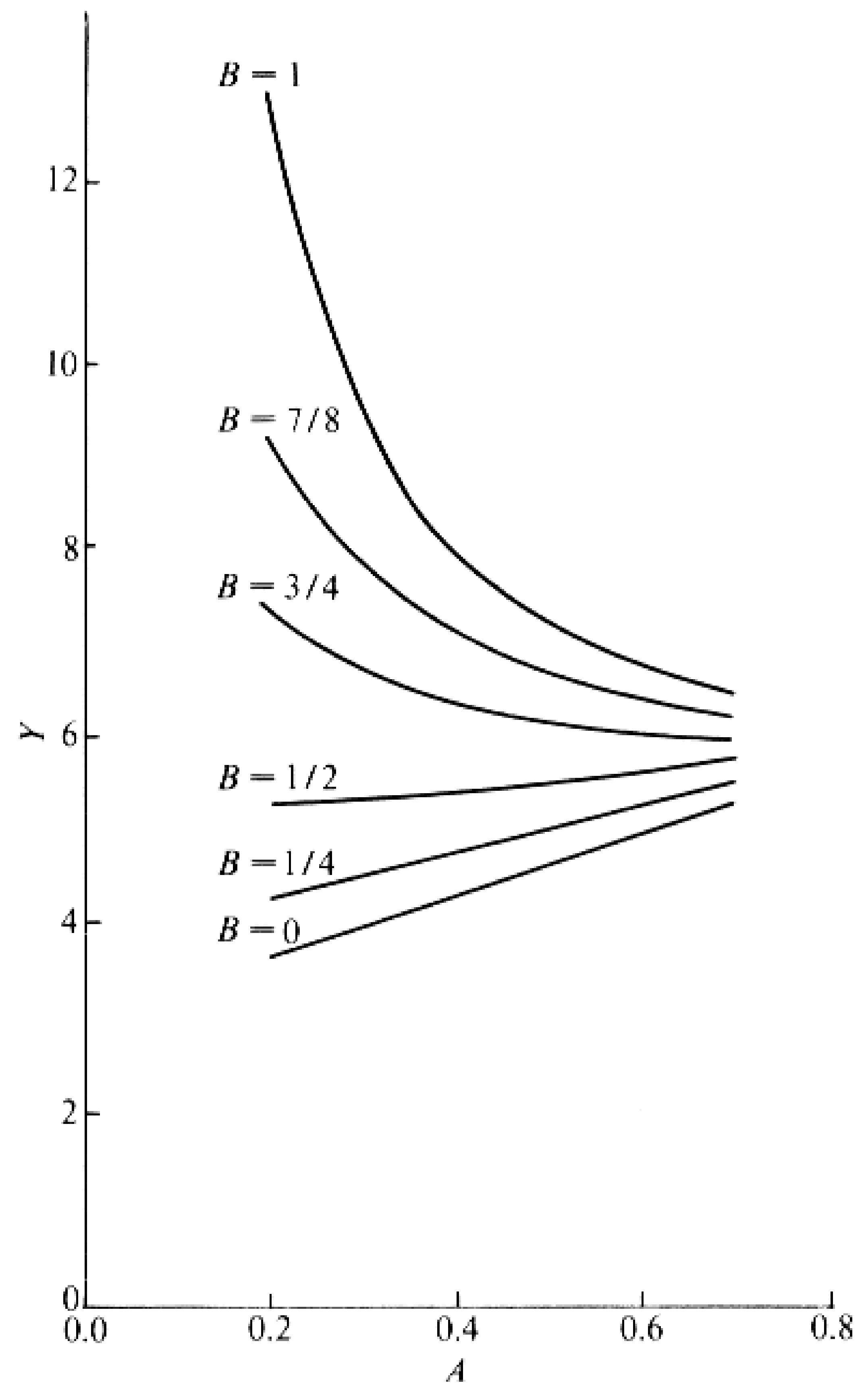

Fig. 81 Auxiliary factor Y (for meanings of symbols see *Pairings*)

(ii) in eqn (63) to obtain n and (iii) in eqn (64) to find out how close the design is to the requirement. From this point it is a matter of adjusting whichever factor is the most convenient for the particular design.

Worked example

It is desired to design a volute spring to fit inside a housing 200 mm in diameter and having a free height of 200–205 mm, a movement of 40 mm under 9000 N, and a total movement from free to solid of 80 mm under 27 000 N, with a maximum stress of 1000 N/mm². Then

(1) $\theta_i = 0.06$;
(2) the estimated value of r_o is 80 mm (half of 200 less space for dead material and clearance);
(3) the estimated value of r_i is 30 mm;
(4) $A = 30/80 = 0.375$;
(5) the ratio of loads is 27 000/9000 = 3, whence from Fig. 79 an estimate of B is 0.5; therefore (from eqn (60)) $0.5 = \{(1 - (0.6/\theta_o)\} \div (1 - 0.375)$, giving $\theta_o = 0.087$;
(6) from Fig. 81, $Y = 5.5$;
(7) from eqn (61), $Z = 0.1 - 0.09 \times 0.375 = 0.066$, and therefore
(8) from eqn (63),

$$1000 = \frac{79\,000t \times 0.06}{0.375 \times 80},$$

whence $t = 6.33$ mm (this is the maximum). The solid height requirement gives $b = 120$ mm to 125 mm, and so $80 = n \times 80 \times 0.06 \times 5.5$, whence $n = 3.03$.

To make a first check, using $b = 125$ mm and $t = 6.33$ mm,

$$W = \frac{79\,000 \times 124 \times 6.33^3 \times 0.06 \times 5.5 \times 0.066}{80^2} = 8523 \text{ N}.$$

The preliminary design is therefore 4½ total coils of 80 mm × 6.5 mm, coiled with an outside diameter of approximately 173 mm (2 × 80 + 2 × 6.33) and an inside diameter of approximately 47 mm (2 × 30 − 2 × 6.33); free height = 125 + 80 = 205 mm, and maximum stress = 79 000 × 0.06 × 6.33/(0.375 × 80) = 1000 N/mm².

This design is approximately 5 per cent weaker than the specification and minor adjustments may be made by the designer to suit stock materials or a stock mandrel. In addition considerable variation in load–deflection characteristics may be effected by tapering the portion of the blade toward either or both ends, and by non-uniform 'pitching' of the coils in their helix.

Applications and materials

Because of the high volume efficiency of this type of spring and the rapid increase in rate at large deflections, which prevents 'bottoming' of the system under shock loads, the volute has found considerable favour with the designer of very heavy vehicles such as tanks, carriers, transporters, and certain classes of railway rolling stock. These springs are of course manufactured from annealed material which is coiled hot and subsequently hardened and tempered, and it is therefore necessary to provide sufficient space between adjoining coils to permit the entry of the quenching oil during the hardening operation. Typical materials would include a plain carbon or one of the low alloy silico-manganese steels.

At the other end of the scale volute springs may be coiled cold from cold-rolled or pre-hardened and tempered stock. Where this is so, it is not necessary to allow for oil penetration between the coils and, if desired, advantage may be taken of the absorption of energy caused by inter-coil friction.

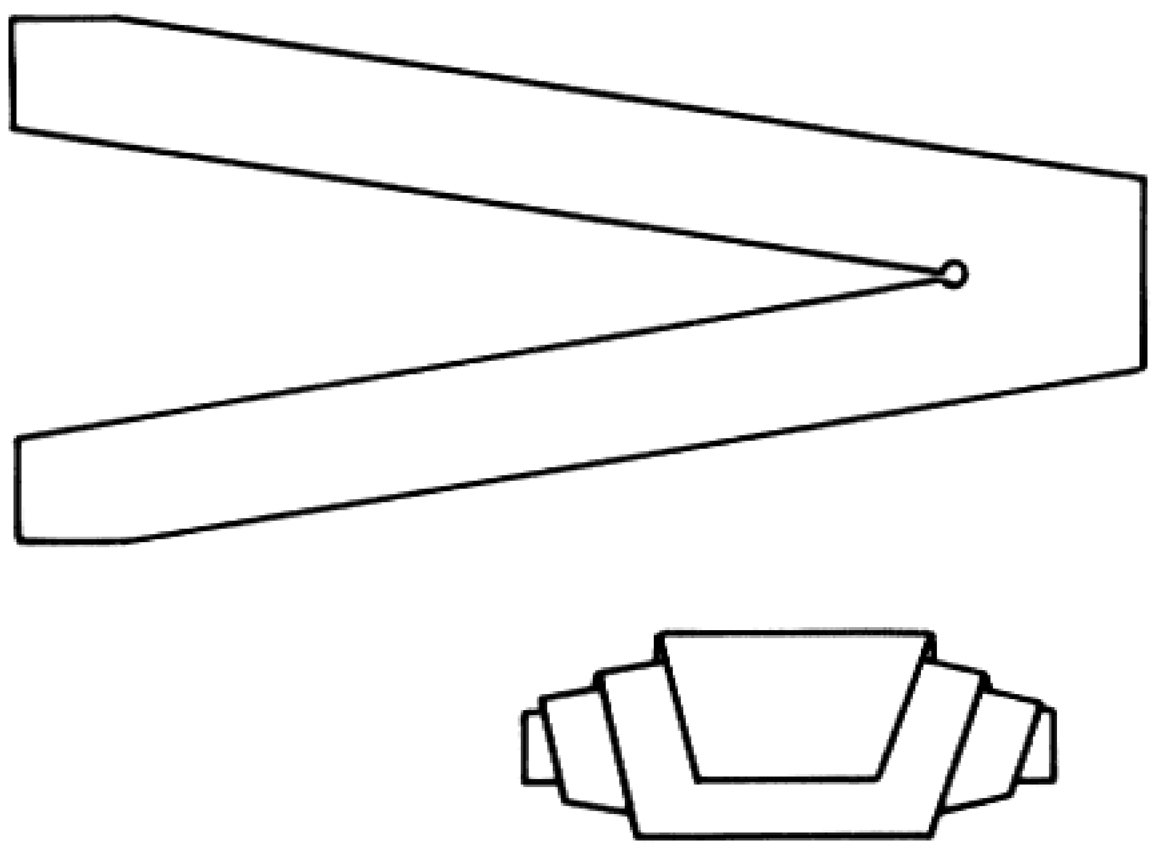

Fig. 82 Secateur spring with developed blank

An interesting example of volute coiled from light gauge material is the well-known 'secateur' spring, which in fact is two volutes coiled base to base from a single piece of strip. The 'blank' from which such a spring is coiled is shown in Fig. 82, the spring being formed by trapping both legs under heels on the same mandrel and coiling to give the familiar double-cone shape.

Appendix. Multileaf torsion bar

Research has been carried out by the author on the possibility of producing a laminated torsion bar, having dimensions well outside the limits specified by previous work in that the angle of twist was to be in the region of 200°. Restrictions on available free length as compared with required torque made it obvious that the number of laminations would be greatly in excess of ten. At the same time it was required that empirical formulae for torsional rate and stress should be developed for springs of this type. Fatigue tests were carried out on a testing machine designed and manufactured by the Spring Research and Manufacturers' Association.

Springs of these dimensions have the disadvantages (1) that the ends tend to splay out during operation and (2) that a hole to accept a locating pin introduces considerable risk of failure from stress concentration. Disadvantage (2) was overcome by omitting the hole and anchoring the pack in a rectangular hole between a pair of hardened pins located on the line of the centre of rotation of the leaves (Fig. 83). The requirement that the leaves should be free to splay and return without undue friction was achieved by allowing space within the mounting for a pair of radiused hardened shims, which were machined to accommodate the tolerance on the pack thickness, that is the cumulative tolerance on a large number of leaves.

The formulae derived were torque,

$$M = \frac{1.45 \times 10^5(n + 0.008n^2) \times \theta(0.017b/t + 0.983) \times b^{1.44} \times t^3}{l^{1.44}} \text{ lb in},$$

where θ is in degrees and dimensions are in inches, and the maximum stress in the outer leaf,

$$f_{max} = \frac{1.2(3b + 2t)}{nt^2b^2} \times t \text{ lb/in}^2,$$

which in 0.7% carbon steel was restricted to 10^5.

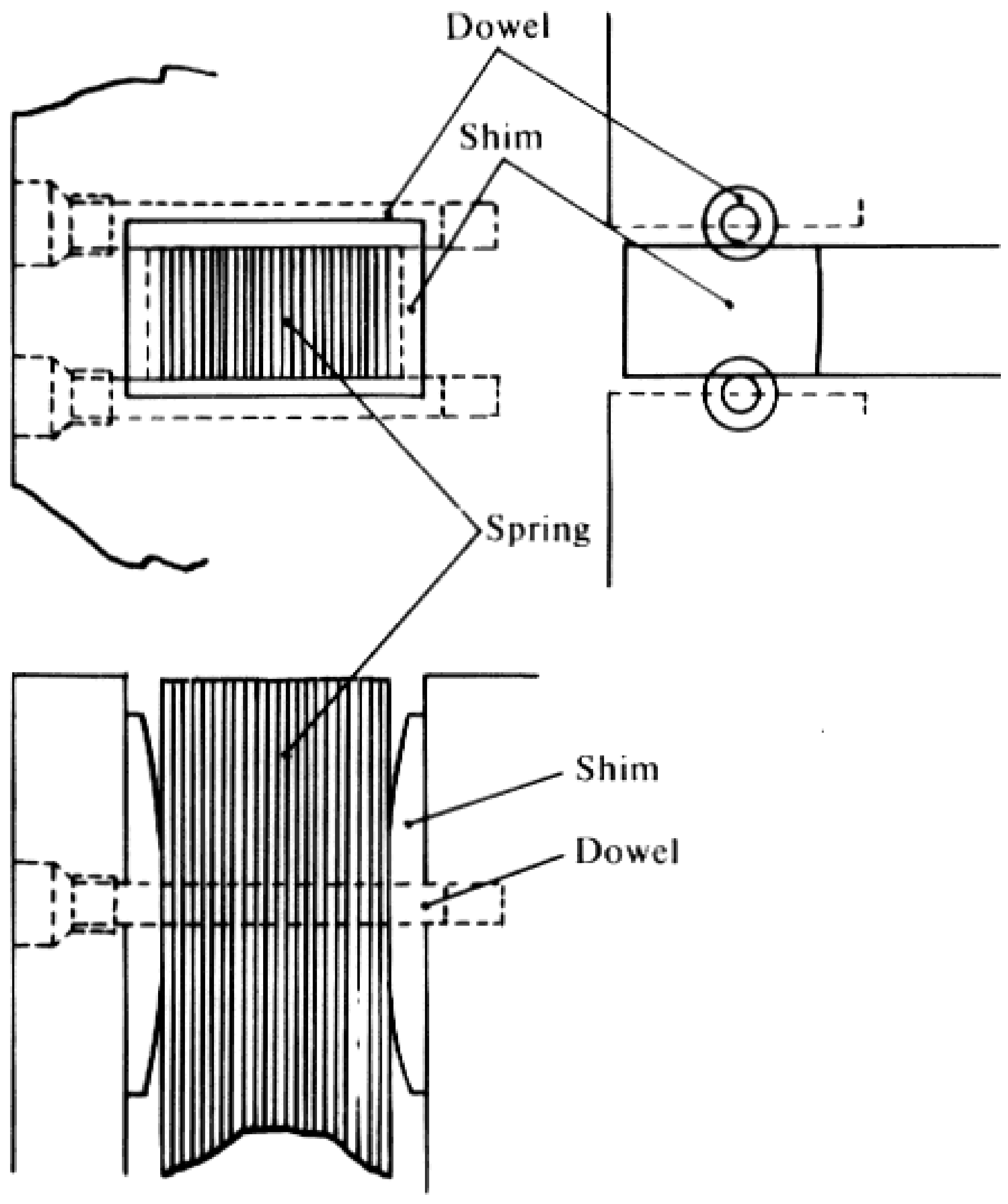

Fig. 83 Schematic assembly of multi-leaf torsion bar

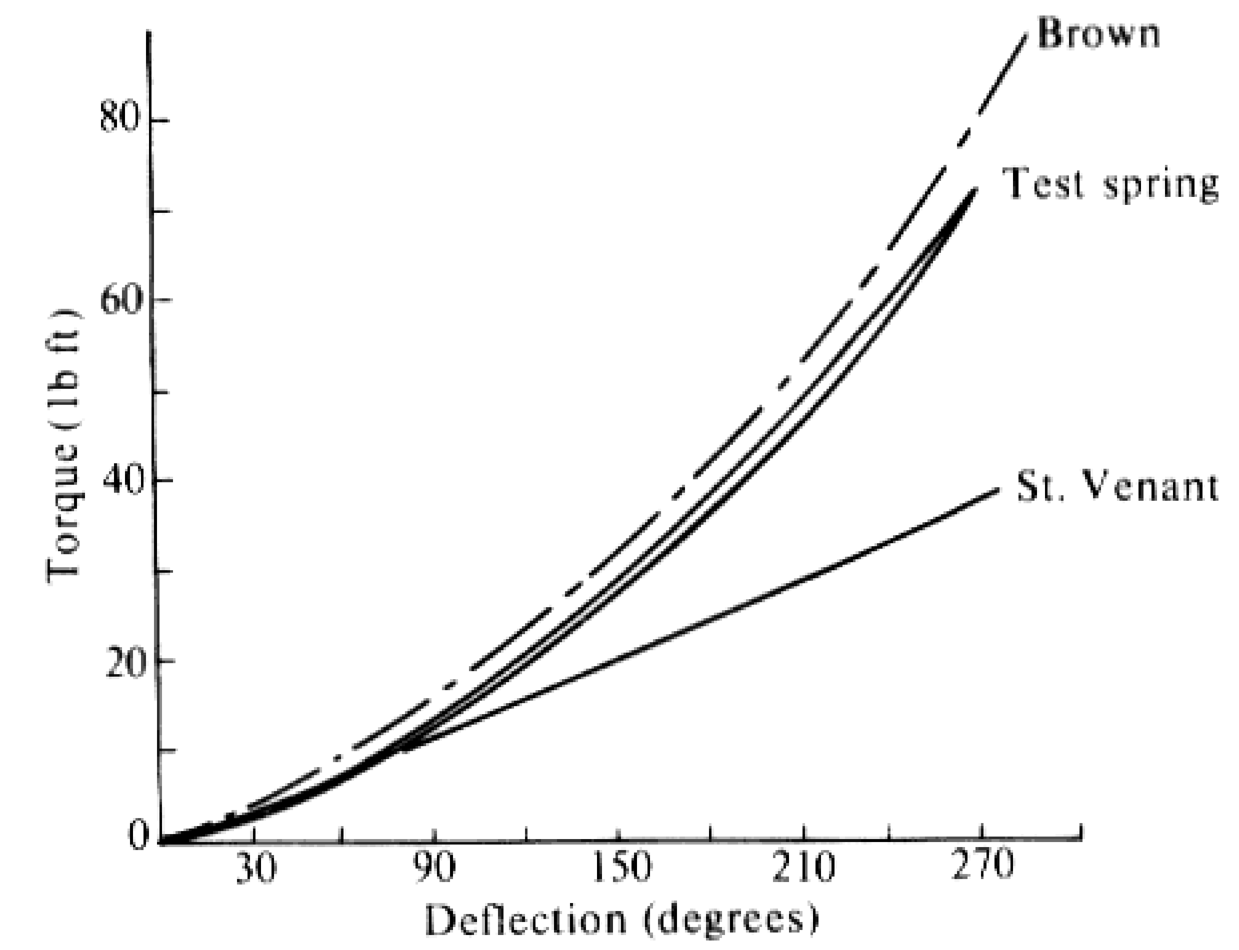

Fig. 84 Comparison of 'Brown' and 'St Venant' factors with spring test results (26 leaves 0.75 in × 0.028 in × 15.35 in)

Although these formulae look forbidding at first sight it is a simple matter to use the St. Venant formulae (eqns (4) and (5)) to produce a tentative design and then modify in the light of Fig. 84 which gives the error (as much as 75 per cent at 240° deflection) between the St. Venant and the Brown curves.

Correlation with practice

Fig. 85 shows the correlation between the derived formula and actual tests carried out on well tried existing designs where the number of leaves, and hence the approximate shape ratio, varies between 18 and 61.

Creep at maximum stress

Fig. 86 shows the result of fatigue tests carried out at a maximum stress of 10^5 lb/in² with a stress range of 4×10^4 lb/in². It will be seen that although the set produced at the end of 10^5 cycles was approximately three per cent, over 90 per cent of this had taken place before 10^4 cycles.

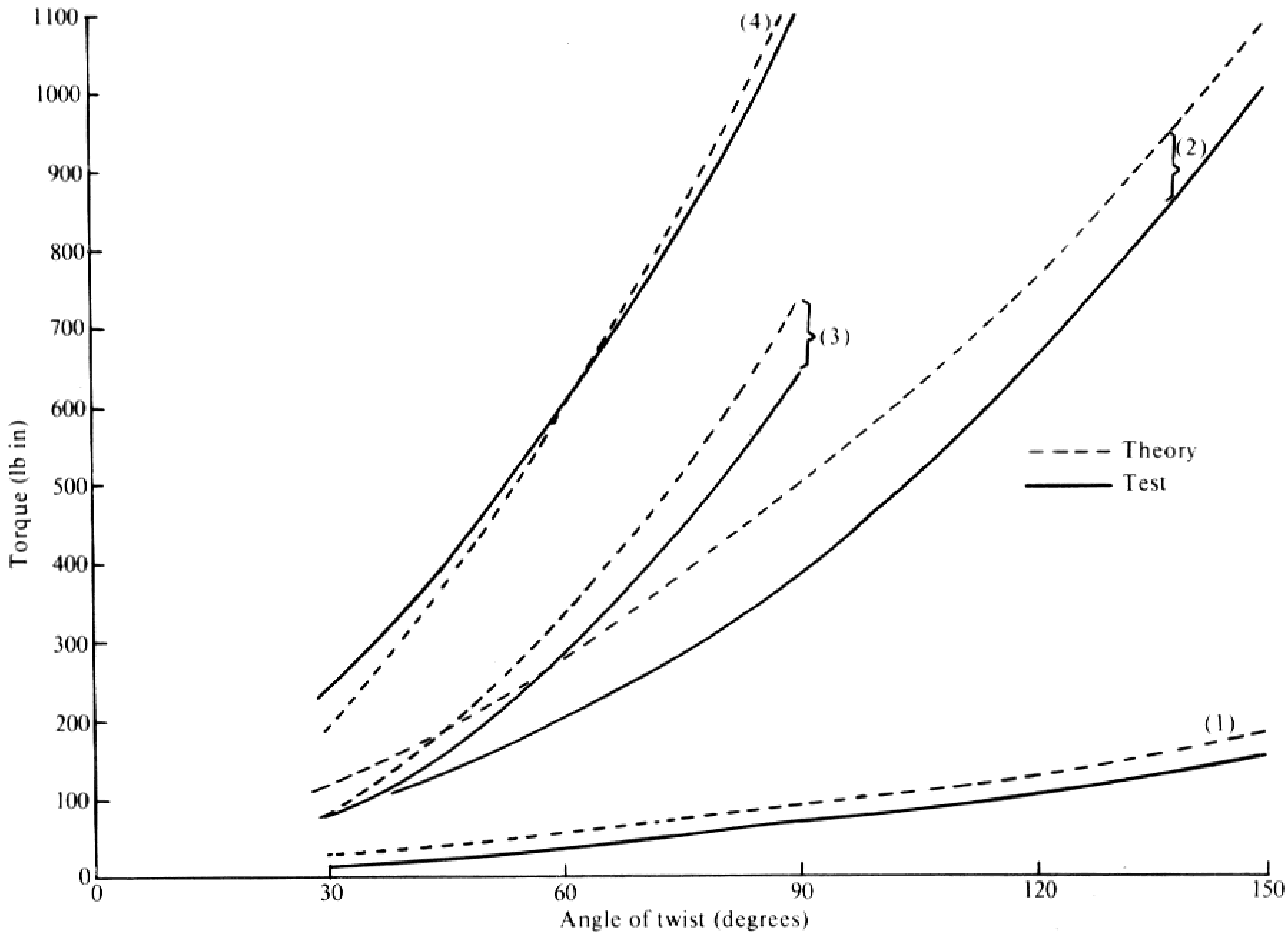

Fig. 85 Spring test results: (1) $n = 20$, $l = 12.5$, $t = 0.02$, $b = 0.75$, $b/t = 37.5$; (2) $n = 24$, $l = 11.17$, $t = 0.028$, $b = 1$, $b/t = 35.7$; (3) $n = 56$, $l = 12.5$, $t = 0.02$, $b = 1.125$, $b/t = 56.25$; (4) $n = 35$, $l = 12.5$, $t = 0.032$, $b = 1.125$, $b/t = 35.16$

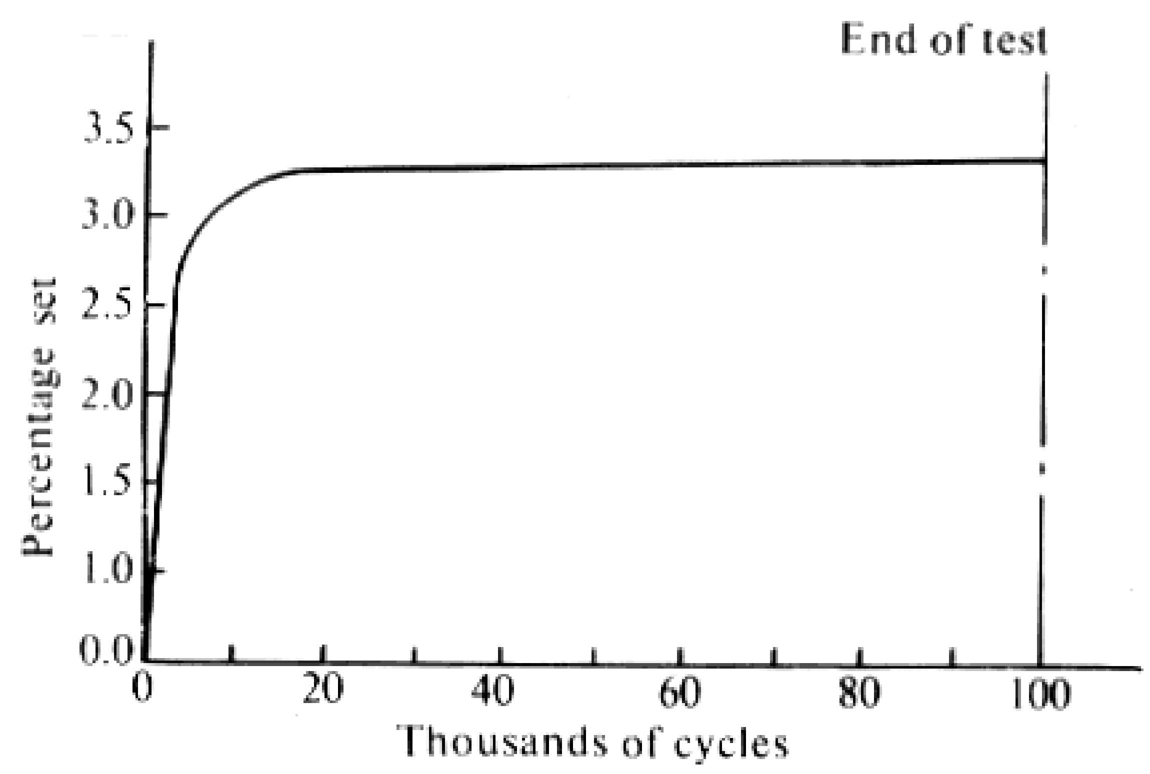

Fig. 86 Percentage set in test spring at 10^5 lb/in^2 during life test

Acknowledgements
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British Standards

BS 970	Wrought steels in the form of blooms, billets, bars and forgings Part 5: 1972 Carbon and alloy spring steels for the manufacture of hot-formed springs
BS 1429: 1948	Annealed steel wire for oil-hardened and tempered springs
BS 1449	Steel plate, sheet and strip Part 1: 1972 Carbon steel plate, sheet and strip
BS 1726	Guide to the design and specification of coil springs Part 1: 1964 Helical compression springs
BS 2506: 1953	Rust-, acid- and heat-resisting steel wire for springs
BS 2786: 1963	Brass wire for springs, 2/1 brass
BS 2803: 1956	Oil-hardened and tempered steel wire for springs for general engineering purposes
BS 2873: 1969	Copper and copper alloys: wire
BS 4391: 1972	Recommendations for metric basic sizes for metal wire
BS 5216: 1975	Patented cold drawn carbon steel wire for mechanical springs
BS AU127: 1966	Road vehicle laminated springs
BS 2TA11: 1974	Titanium-aluminium-vanadium alloy bars for machining
BS 2TA12: 1974	Titanium-aluminium-vanadium alloy forging stock

British Standards may be obtained from BSI, Sales Department, 101 Pentonville Road, London N1 9ND.

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†Spring Research and Manufacturers' Association (1975). Design charts for helical torsion springs. *Spring design data sheet* No. 11. Sheffield.
——— (1968). *Springs—materials, design, manufacture.* Sheffield.
——— (1974). Helical springs. *Engineering Design Guide* 08. Oxford University Press.

†This book is available from The Spring Research and Manufacturers' Association, Henry Street, Sheffield S3 7EQ.

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Prior to his retirement in 1977, Mr. Brown was Chief Research Scientist with the Geo. Salter Group of Companies. He is now a consultant to The Spring Research and Manufacturers' Association.

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